Monads Are *Not* About Sequencing

Lawful Monads in a Concurrent Setting

functional scala

Tomas Mikula

Dec 1st, 2023

“Monads are about sequencing.”

–folk knowledge

What Does *Sequencing* Mean?

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Spoiler

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Spoiler

¿ Sequence-like syntax ? ⬅ the only defensible claim

¿ Enforced sequential execution ?

¿ Enforced order of “effects” ?

Monads: Refresher

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… plus the monad laws …

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given M: Monad[M]

val ma: M[A]

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M[M[M[M[D]]]]
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Right?

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Subtype Relation <:< Forms a Category

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given Category[<:<] with

    def andThen[A, B, C](f: A <:< B, g: B <:< C): A <:< C  =  f andThen g

    def id[A]:                                    A <:< A  =  <:<.refl
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Monads in <:<

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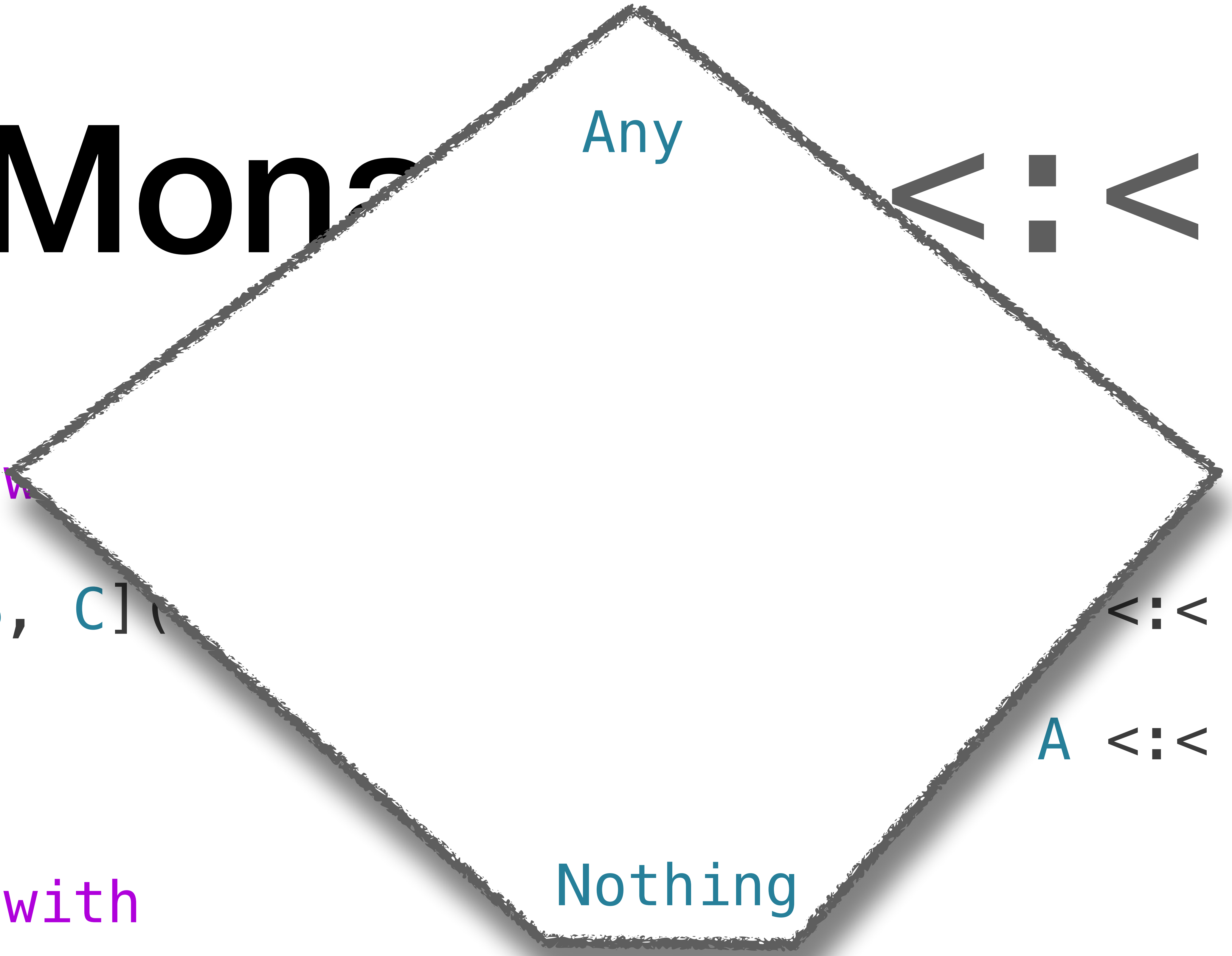
Monads in the category `<:<` are **Closure Operators.**

Mon… <:<

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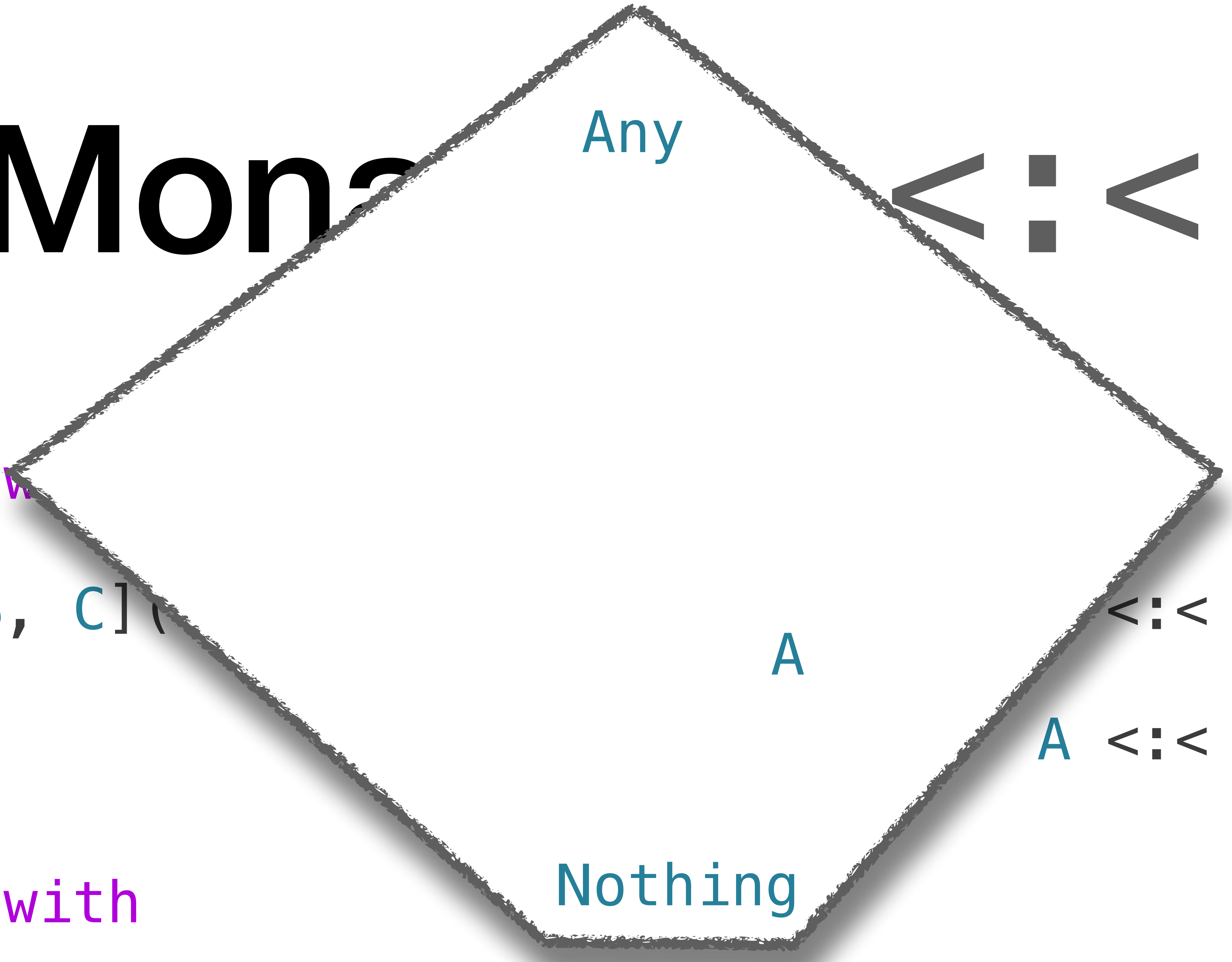


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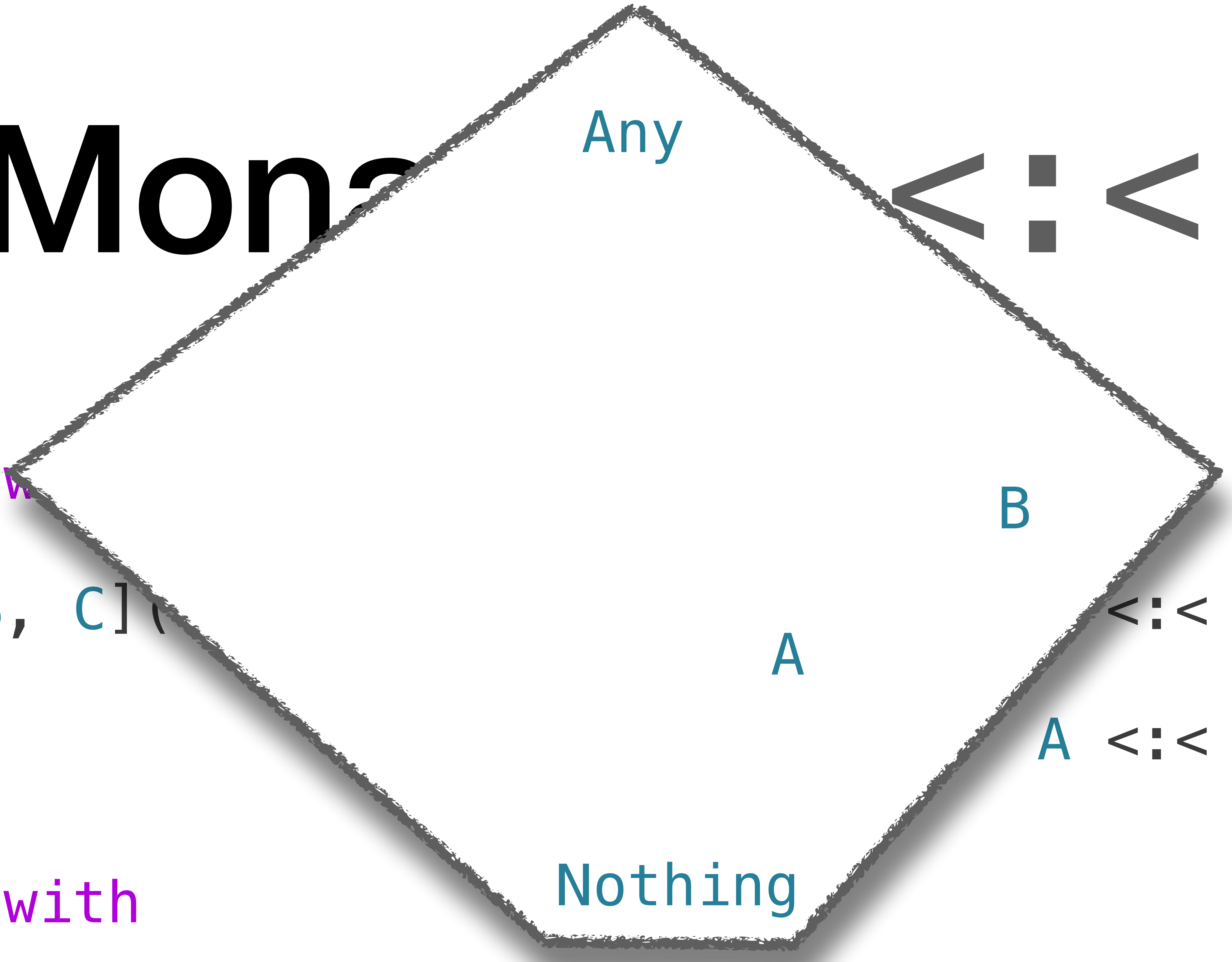


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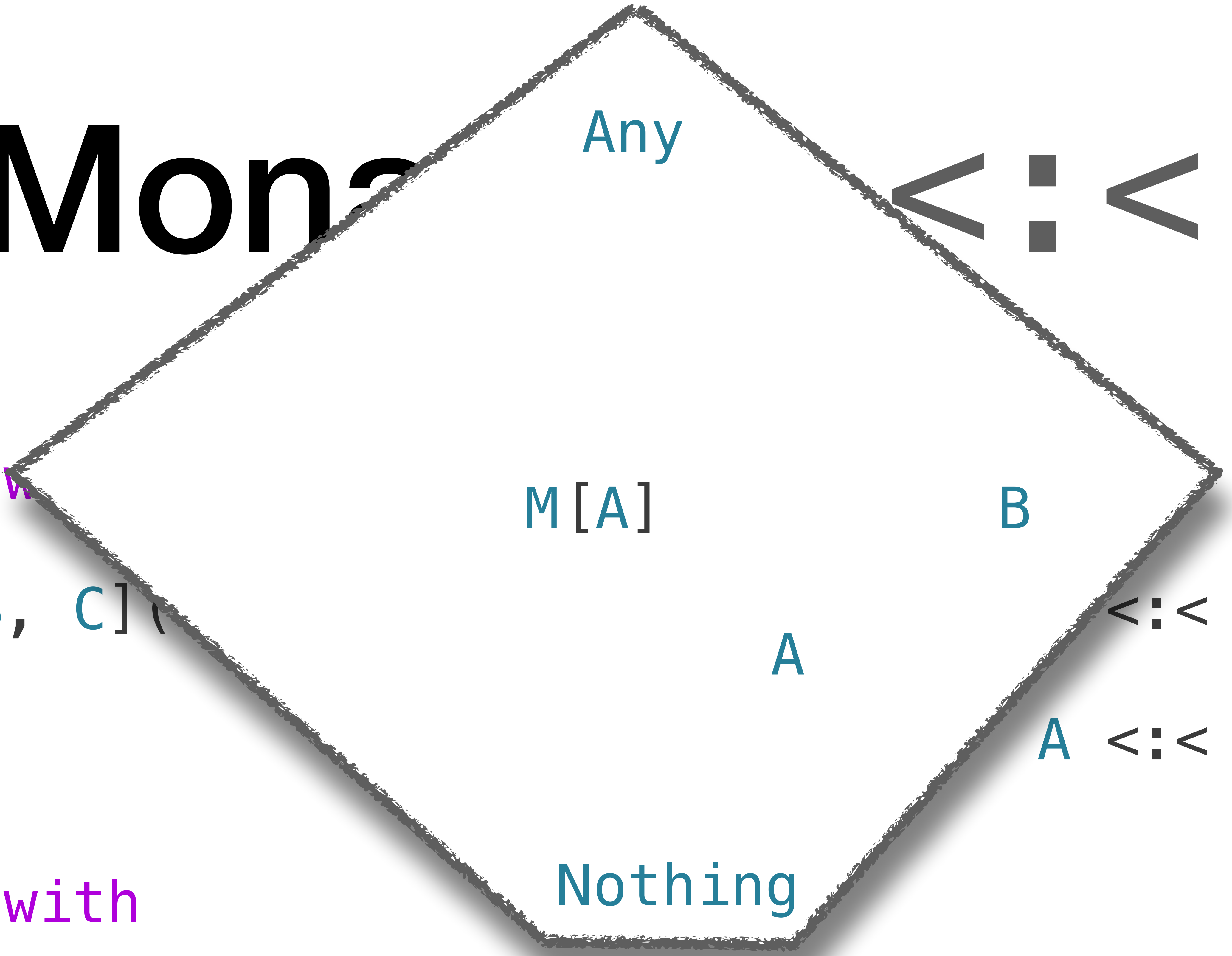


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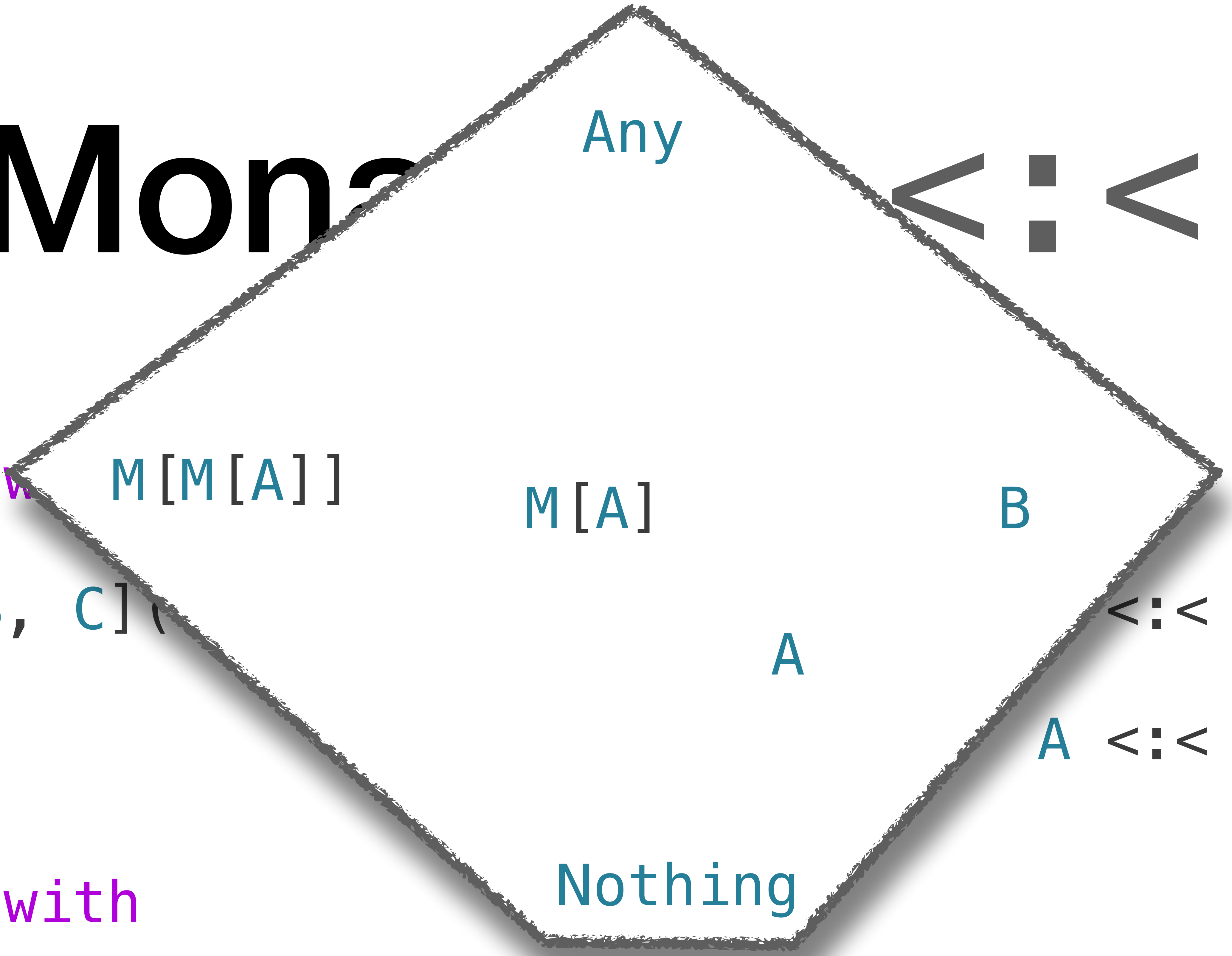


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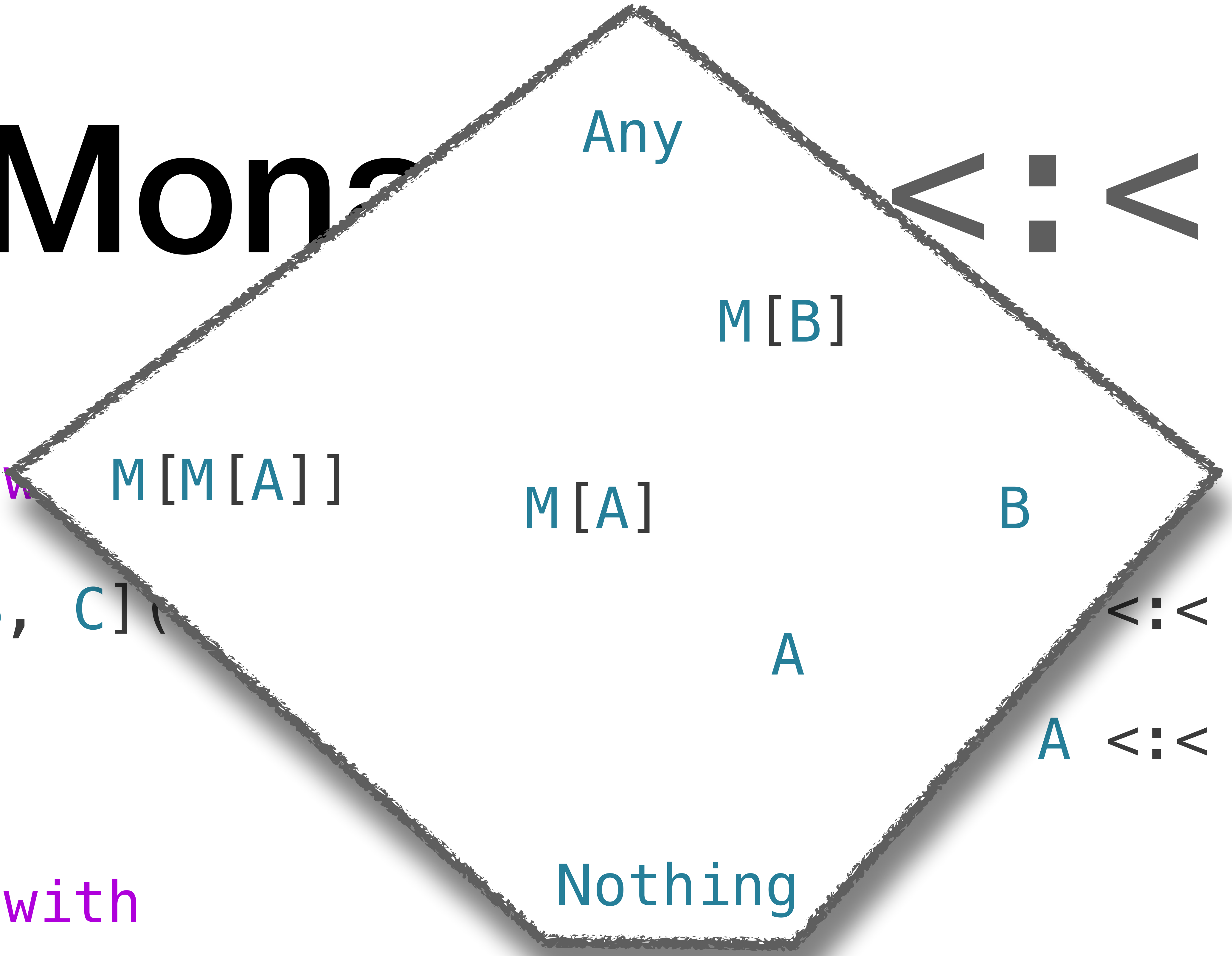


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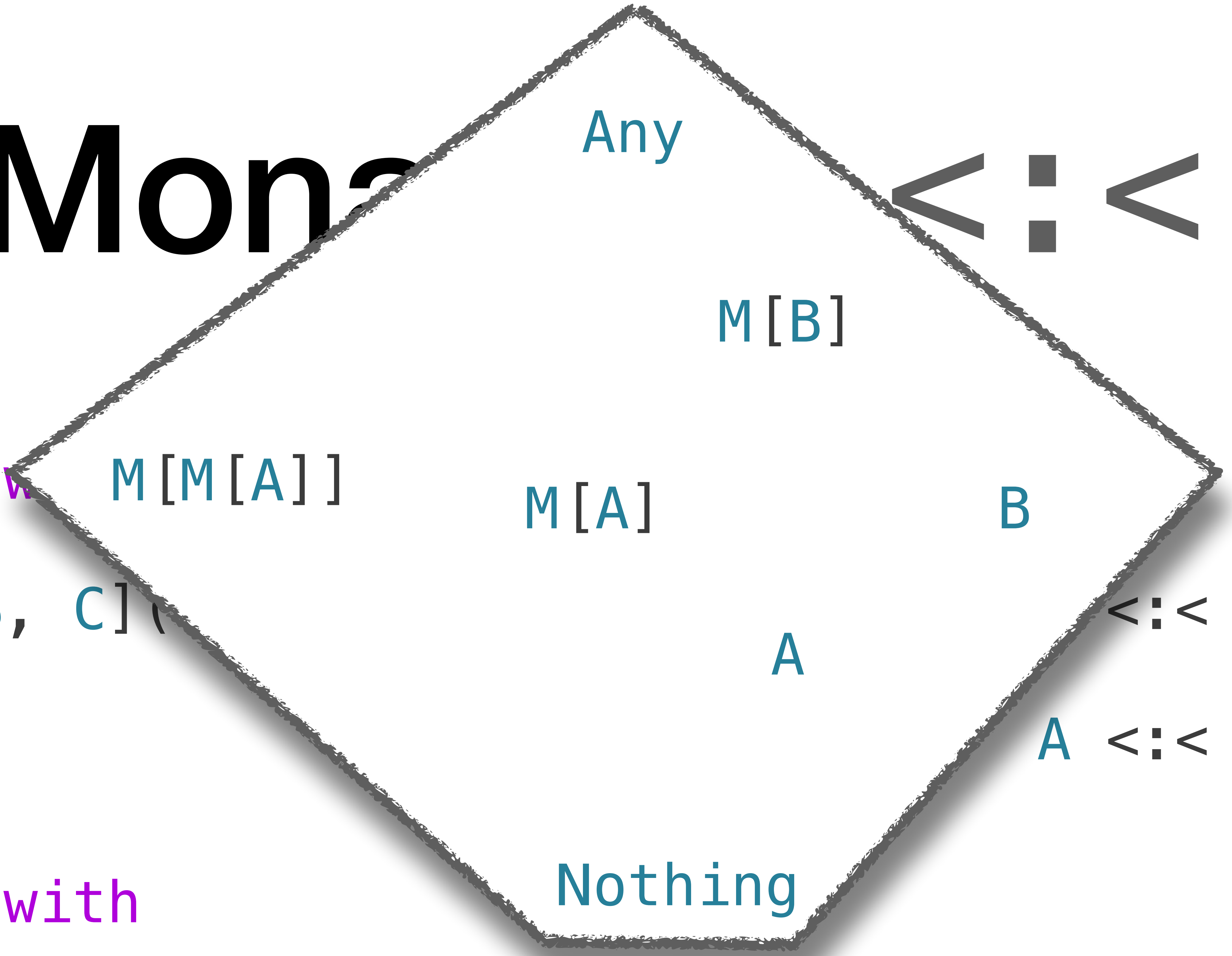


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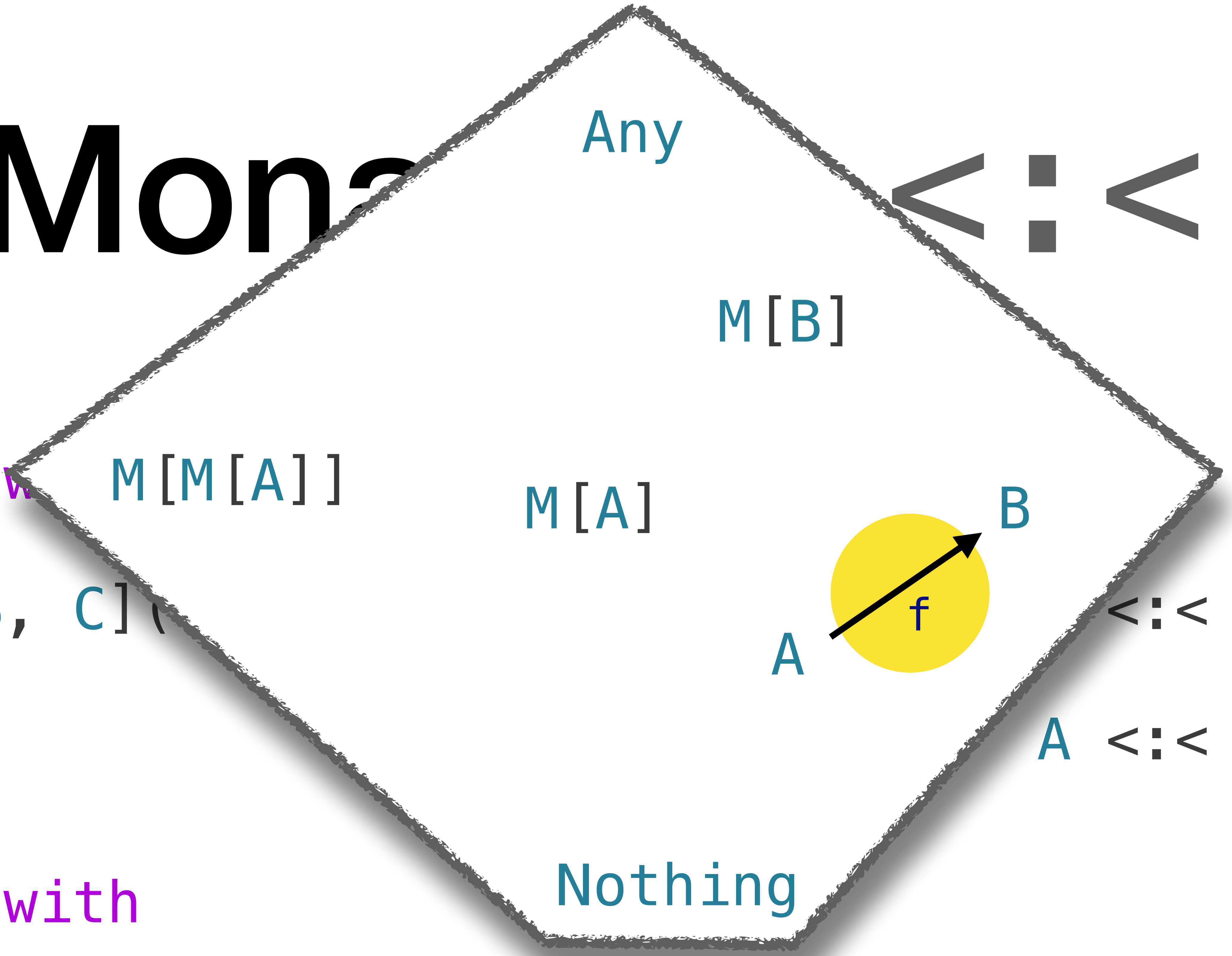


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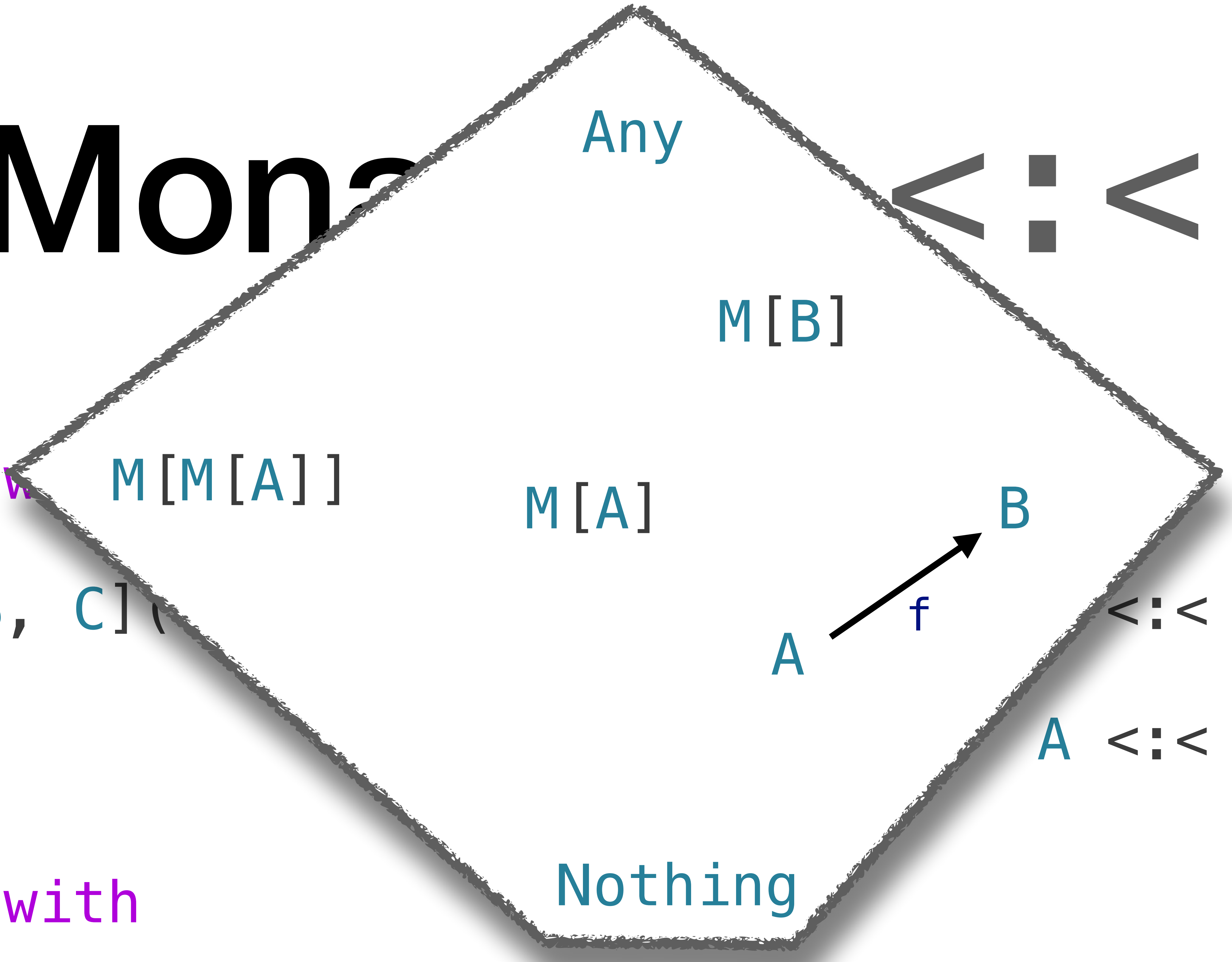


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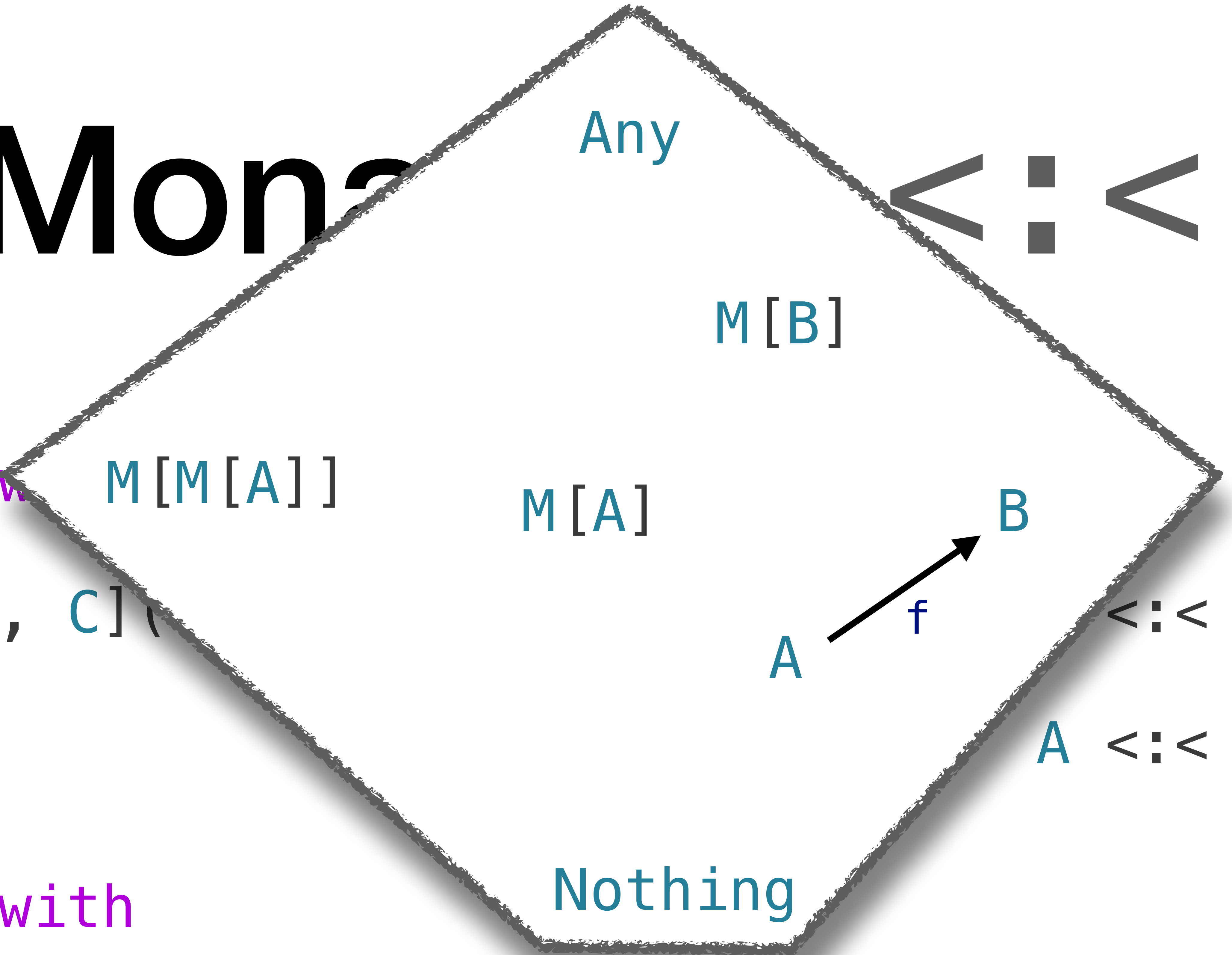


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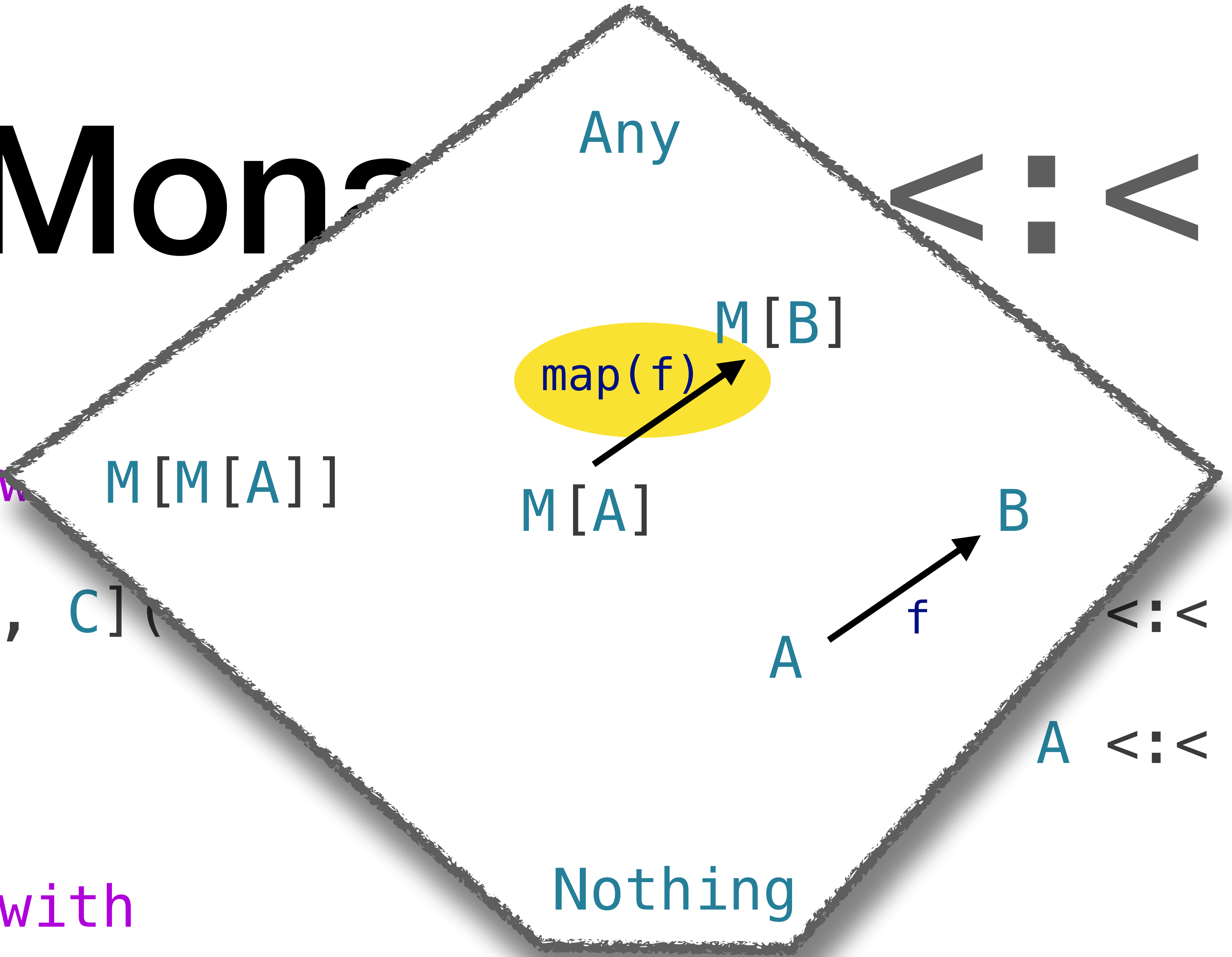


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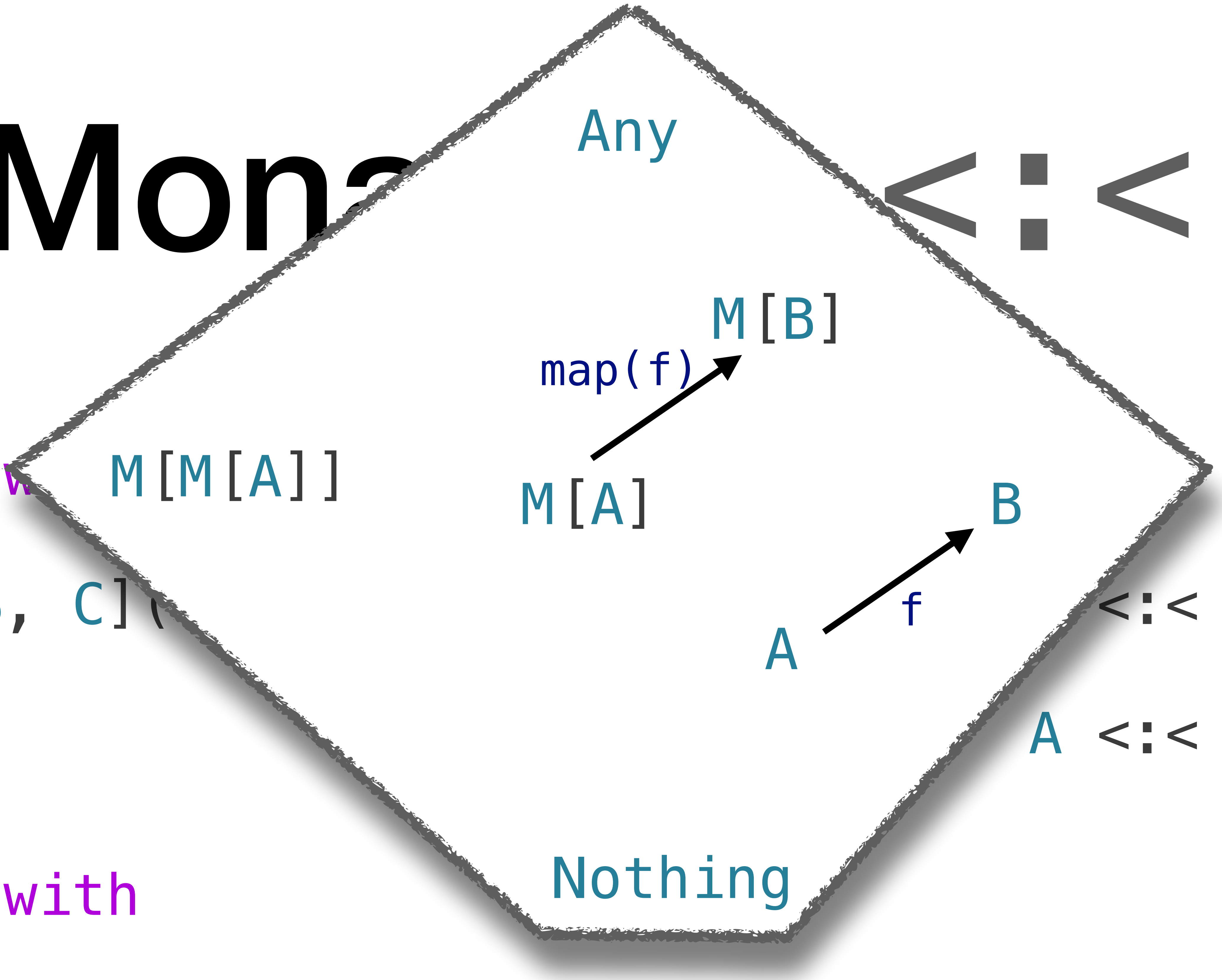


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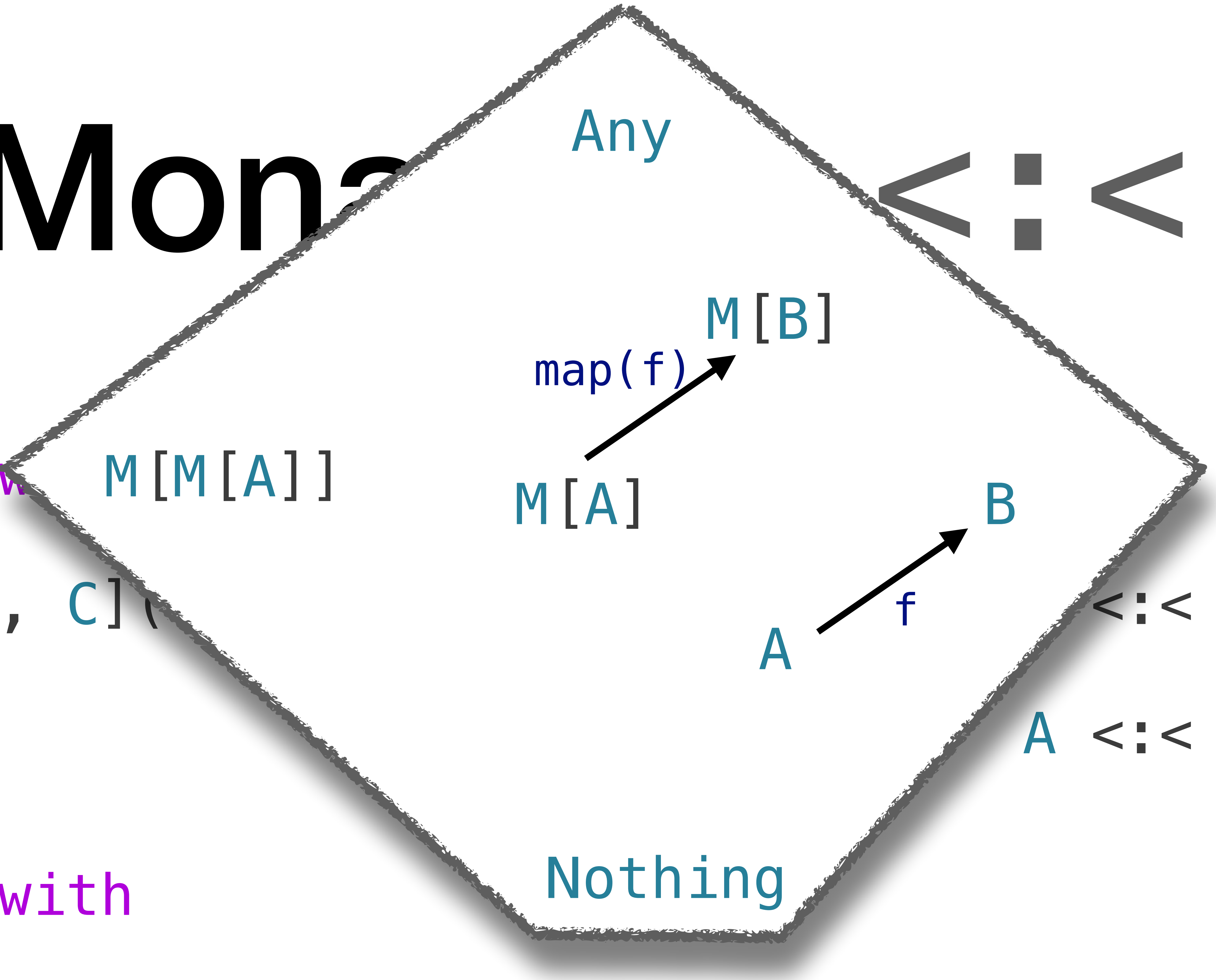


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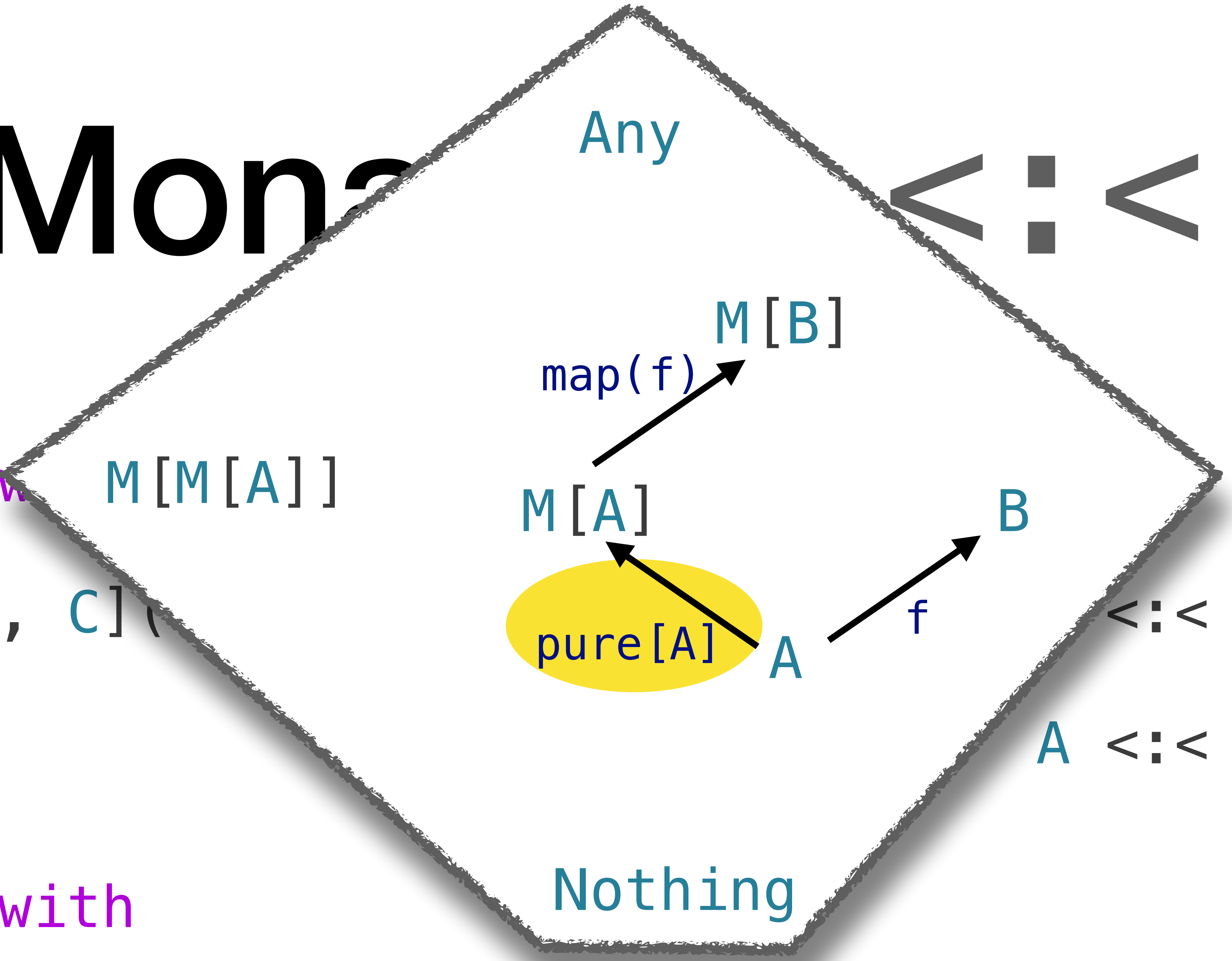


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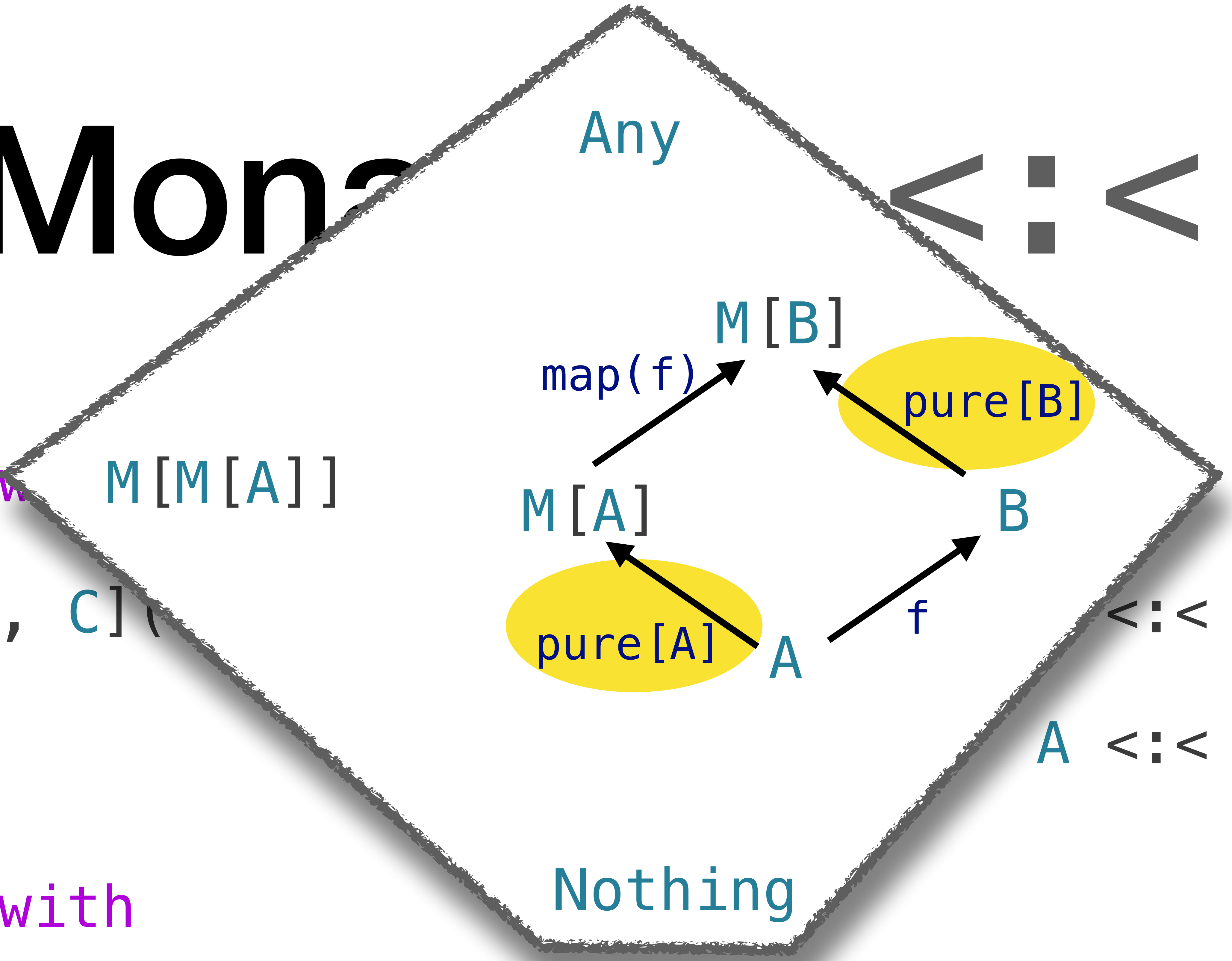


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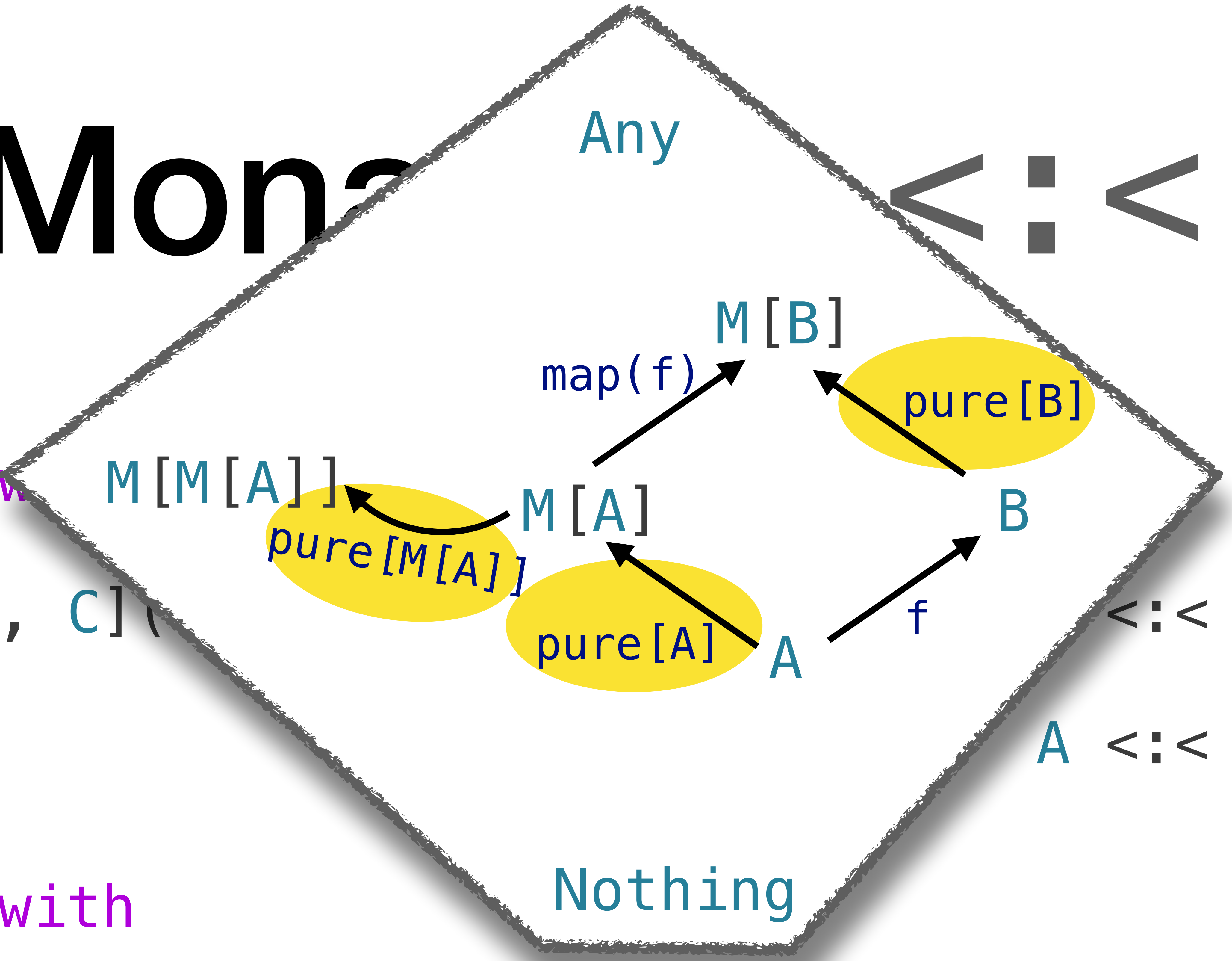


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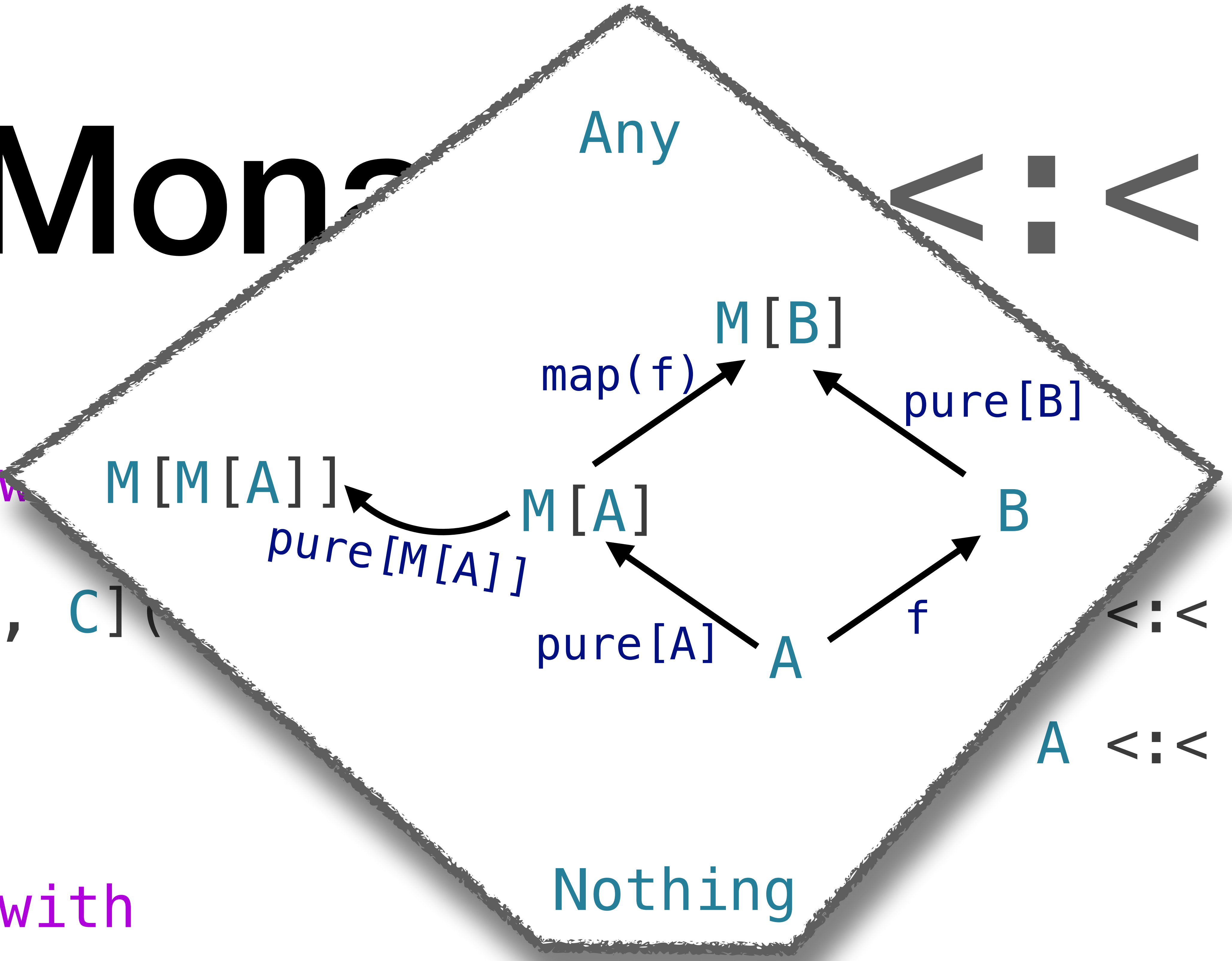


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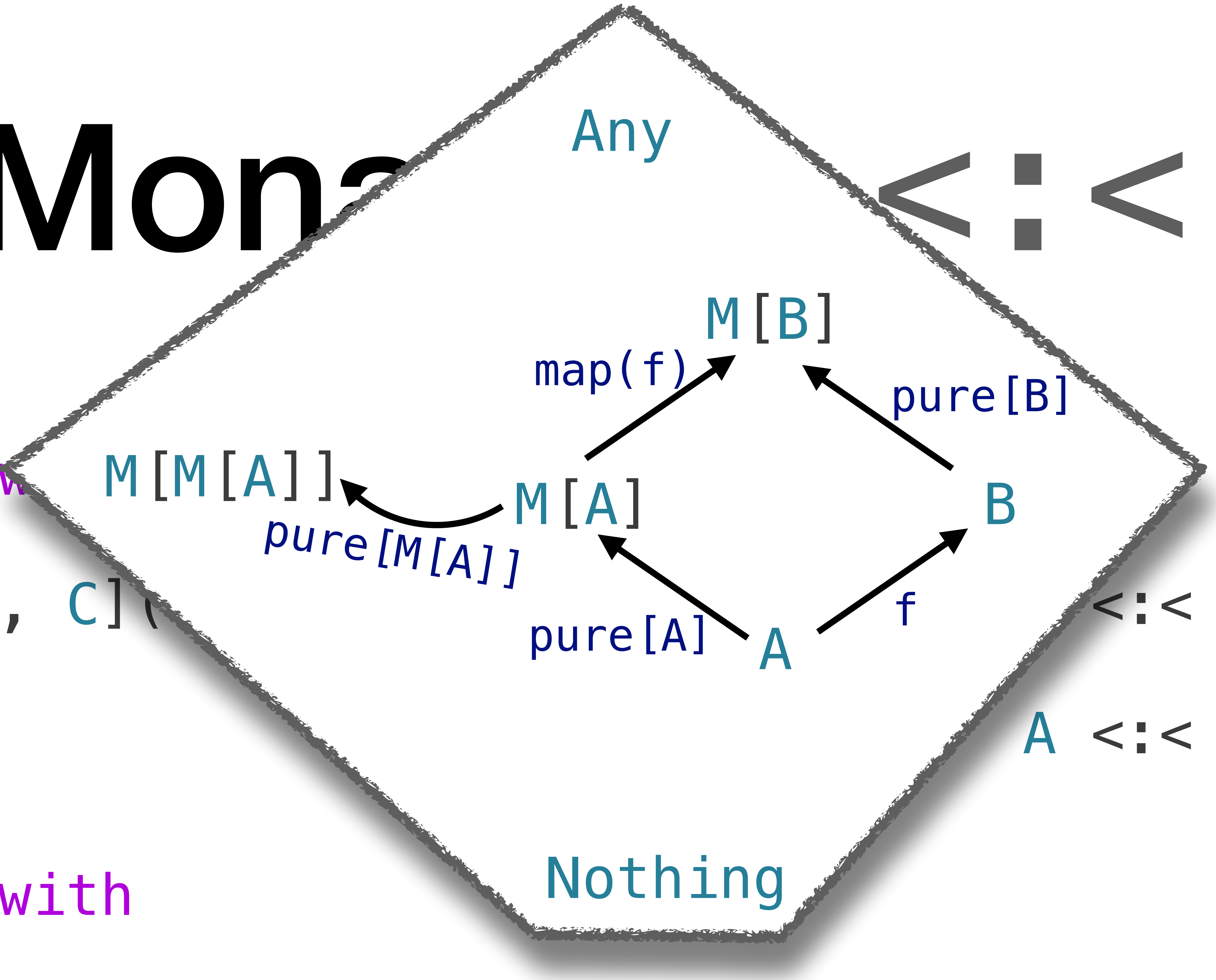


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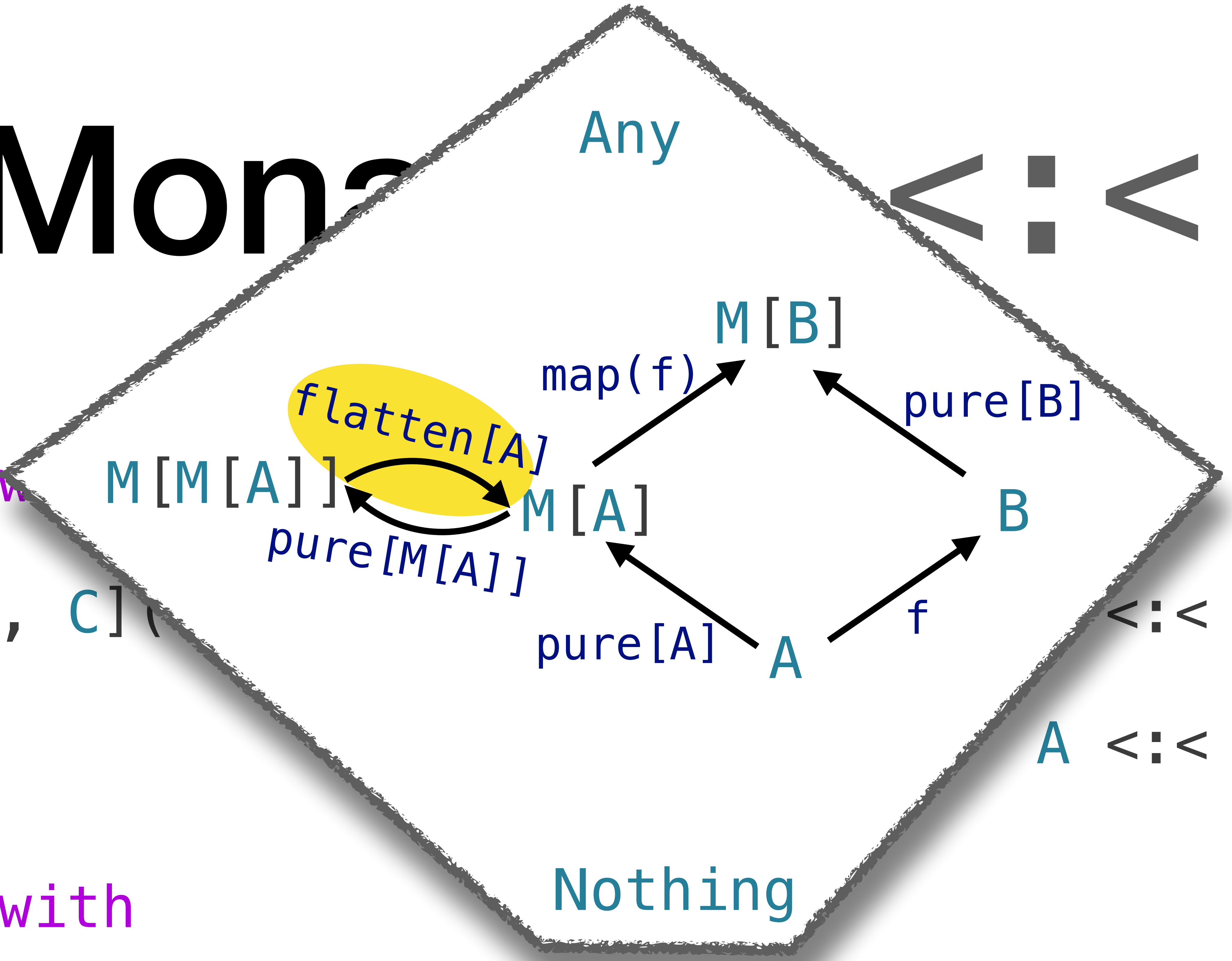


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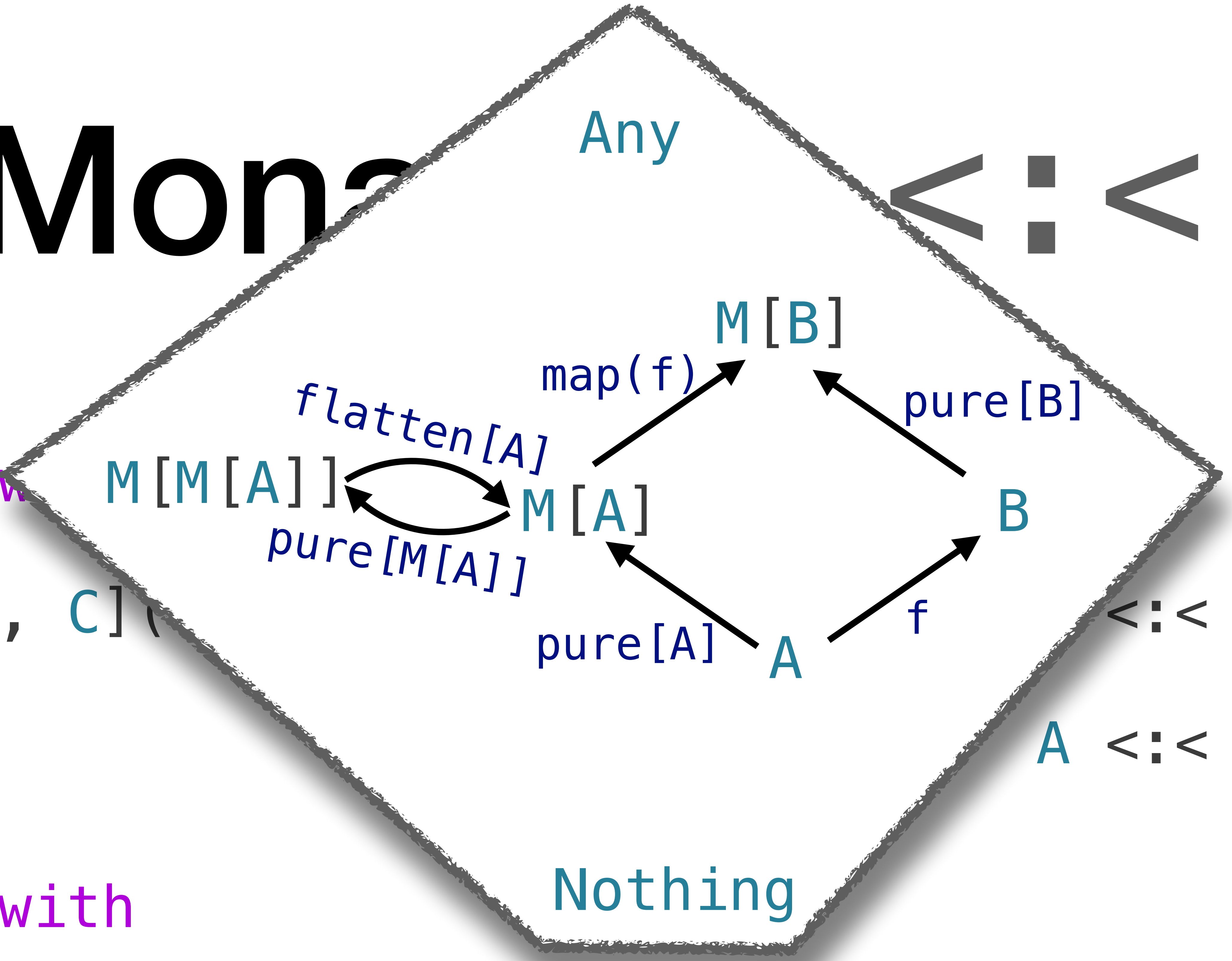


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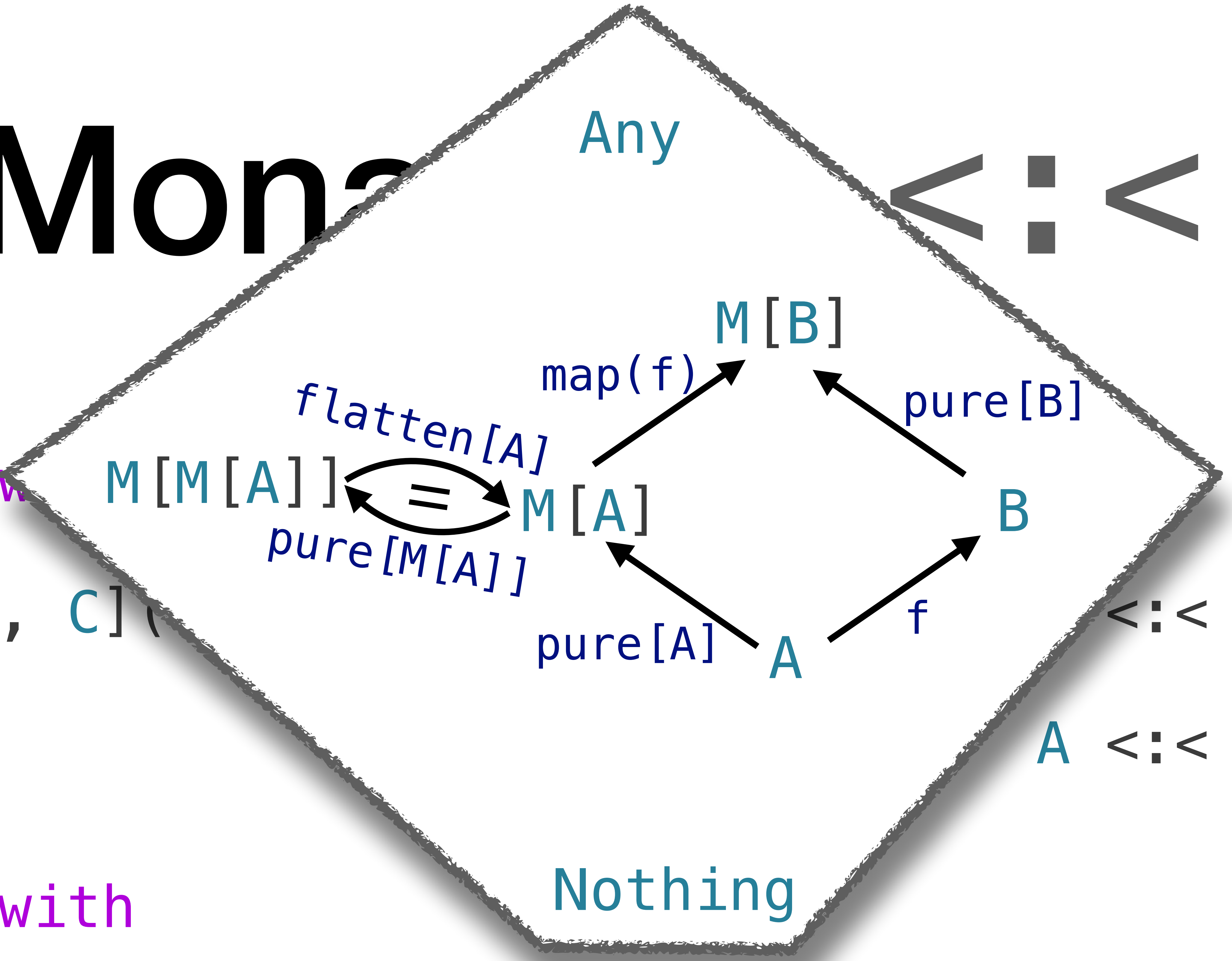


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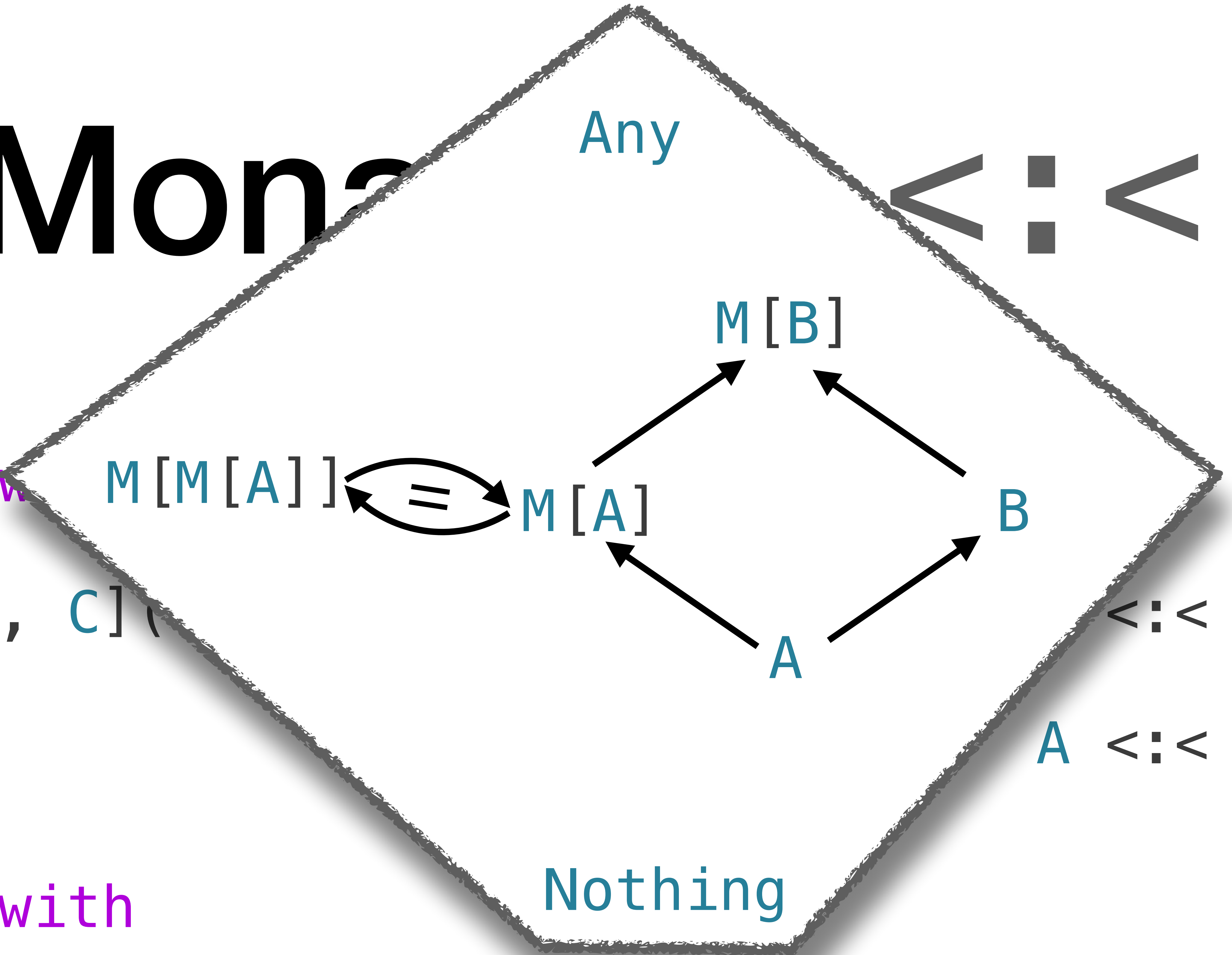


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Monads in the category <:< are **Closure Operators.**

Monads in <:< : Example

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type Res[+A] = Error | A

given Monad[<:<, Res] with

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        ???

    def pure[A] : A <:< Res[A] =

        ???

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Monads in <:< : Example

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given Monad[<:<, Res] with

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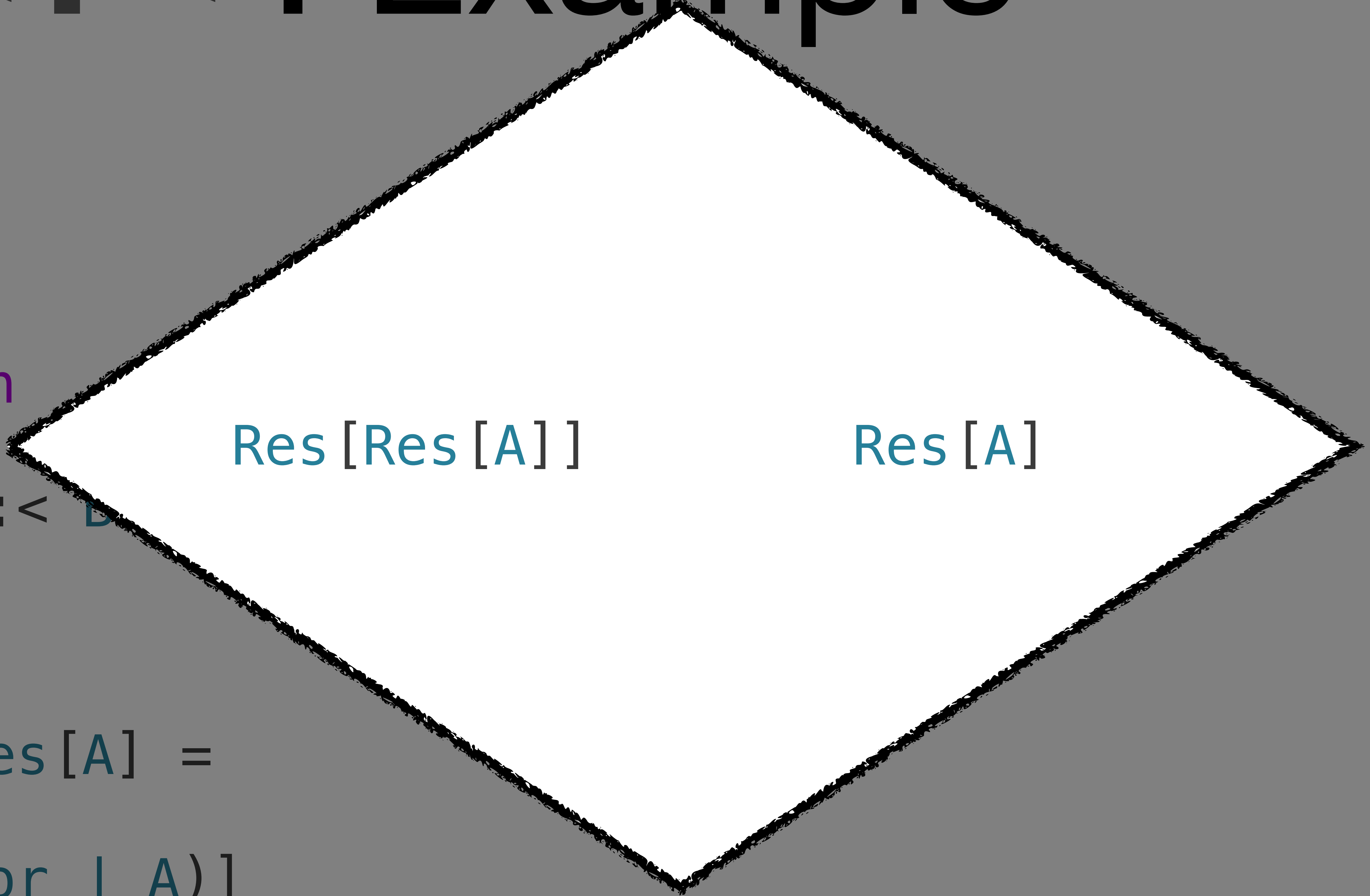


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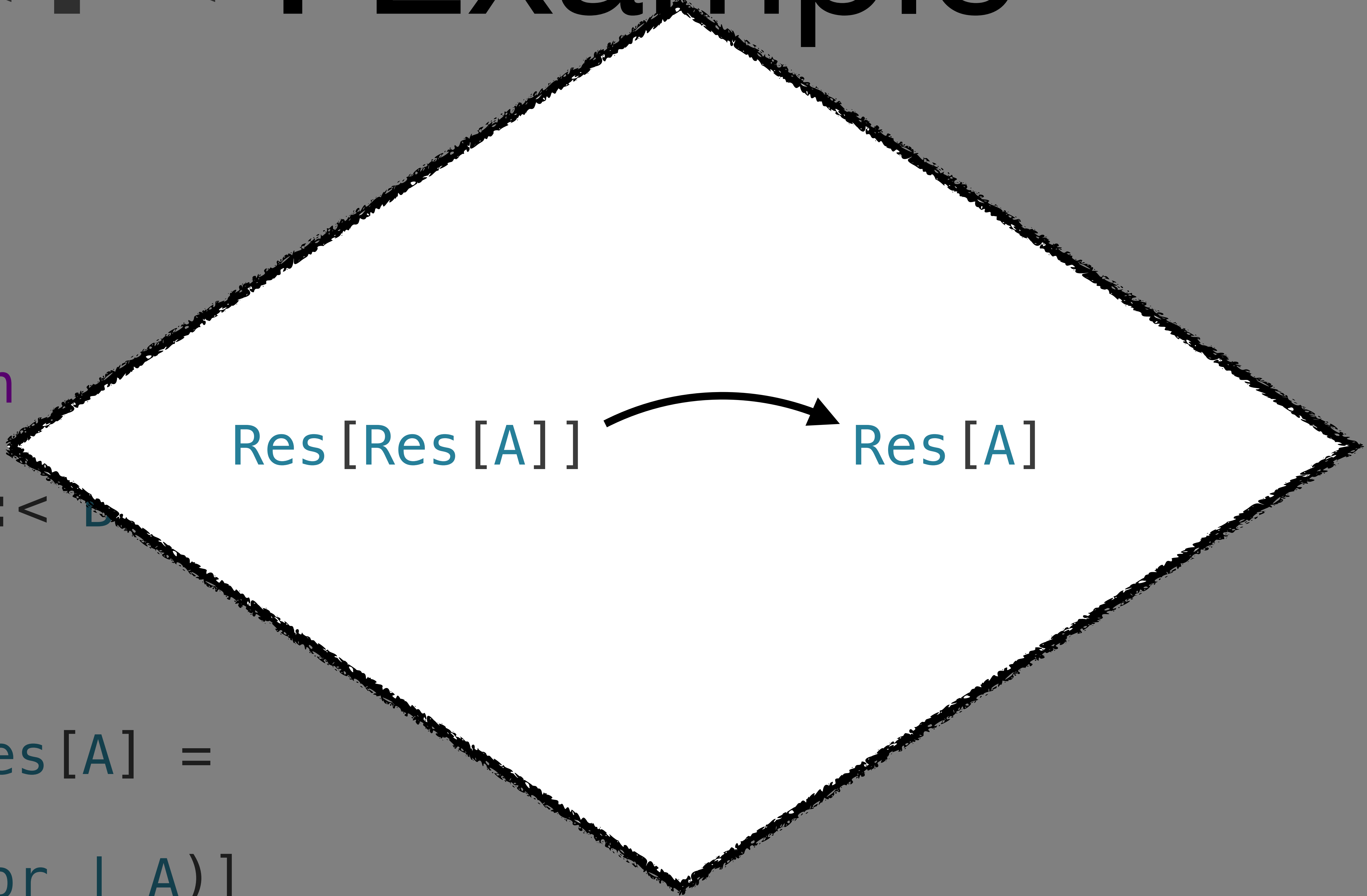


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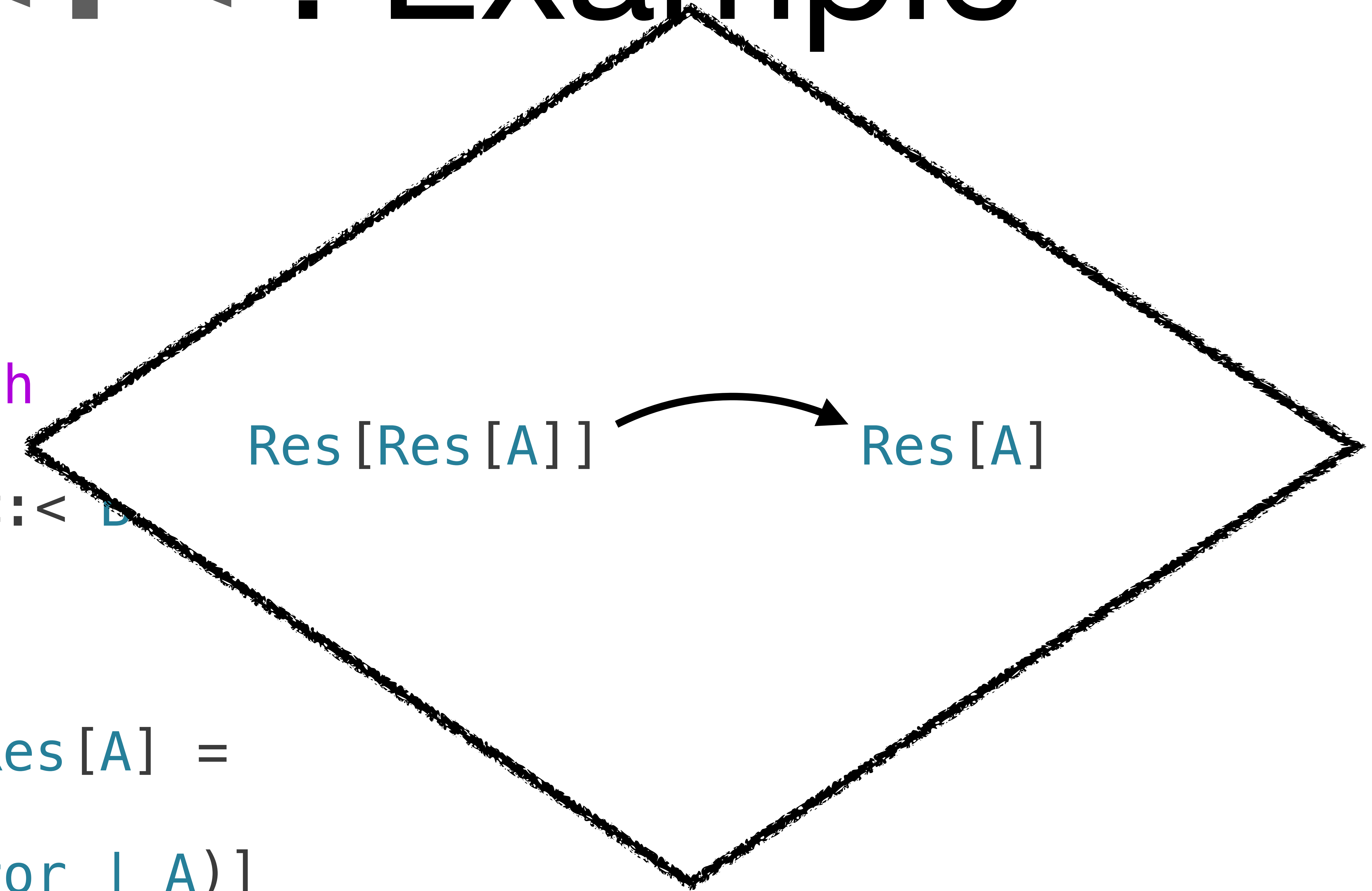

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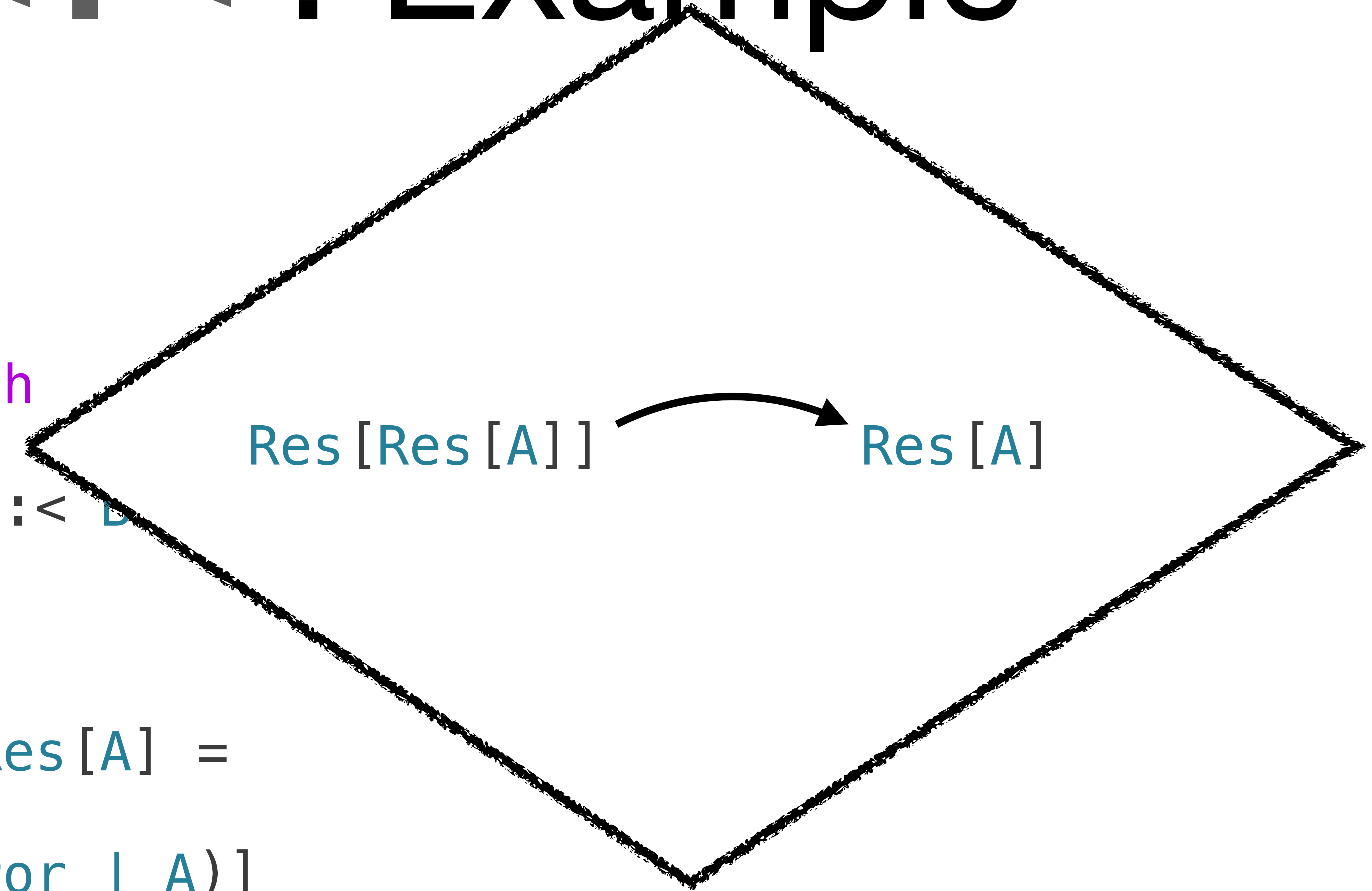

Sequencing? Not even execution! Just statements about types.

Lesson So Far

- Monads definable in **any** Category (even non-executable one, like `<:<`)

Game Over

GAME OVER

EXIT

GAME OVER

EXIT

PLAY AGAIN

GAME OVER

EXIT

PLAY AGAIN

The Opposite Category

```
case class Op[A, B](run: B => A)
```

The Opposite Category

```scala
case class Op[A, B](run: B => A)

type <= [A, B] = Op[A, B]
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case class Op[A, B](run: B => A)

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given Category[<=] with
```

The Opposite Category

```scala
case class Op[A, B](run: B => A)

type <= [A, B] = Op[A, B]

given Category[<=] with
    def andThen[A, B, C](f: A <= B, g: B <= C): A <= C =
```

The Opposite Category

```scala
case class Op[A, B](run: B => A)

type <= [A, B] = Op[A, B]

given Category[<=] with
    def andThen[A, B, C](f: A <= B, g: B <= C): A <= C =
        Op(g.run andThen f.run)
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The Opposite Category

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        Op(g.run andThen f.run)
    def id[A]: A <= A =
        Op(a => a)
```

Monads in <=

Monads in <=

Monads in **<=** are exactly the *co-monads* in **=>**.

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Example: The **`Id`** (co-)monad

```
case class Id[A](a: A)
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Example: The **`Id`** (co-)monad

```
case class Id[A](a: A)


given Monad[<=, Id] with
```

Monads in <=

Monads in **<=** are exactly the *co-monads* in **=>**.

Example: The **Id** (co-)monad

```
case class Id[A](a: A)

given Monad[<=, Id] with
    def map[A, B](f: A <= B): Id[A] <= Id[B] =
```

Monads in <=

Monads in **<=** are exactly the *co-monads* in **=>**.

Example: The **`Id`** (co-)monad

```
case class Id[A](a: A)

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Monads in <=

Monads in **<=** are exactly the *co-monads* in **=>**.

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case class Id[A](a: A)

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  def map[A, B](f: A <= B): Id[A] <= Id[B] =
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  def pure[A]: A <= Id[A] =
```

Monads in <=

Monads in **<=** are exactly the *co-monads* in **=>**.

Example: The **Id** (co-)monad

```
case class Id[A](a: A)

given Monad[<=, Id] with
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Monads in **<=** are exactly the *co-monads* in **=>**.

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val f: String  <= Id[Boolean]   = Op { case Id(b)  => println("f"); String.valueOf(b) }
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Full code at https://github.com/TomasMikula/non-sequencing-monads/

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    i <- g(b)
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yield l)
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Y U NO UNDERSTAND

IT’S THE **EFFECTS** THAT ARE SEQUENCED

“`flatten` sequences the *effects*.”

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What?

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Just reject this game. 🙅‍♂️

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Writer Monad with a Twist

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type Writer[A] = (List[String], A)
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`Writer[A]` is a lawful *Monad* for **any** lawful *Monoid* on `List[String]`.

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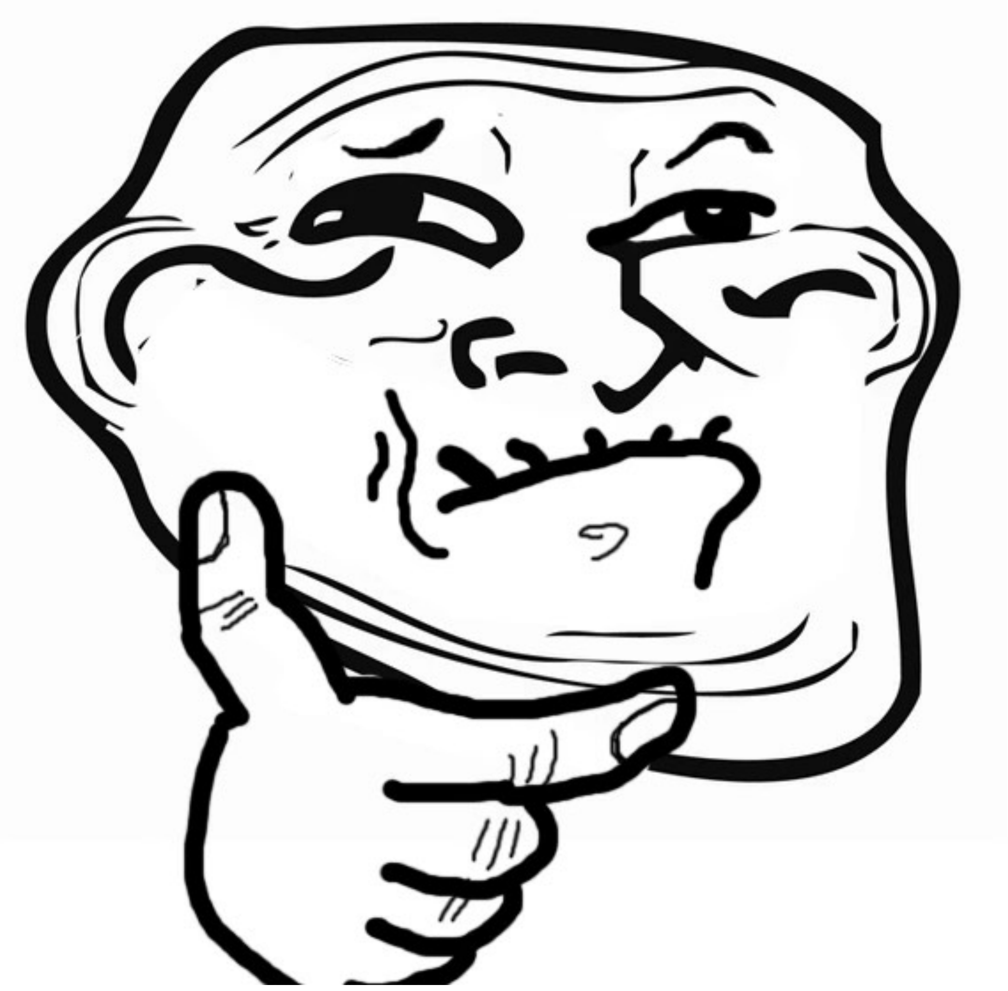
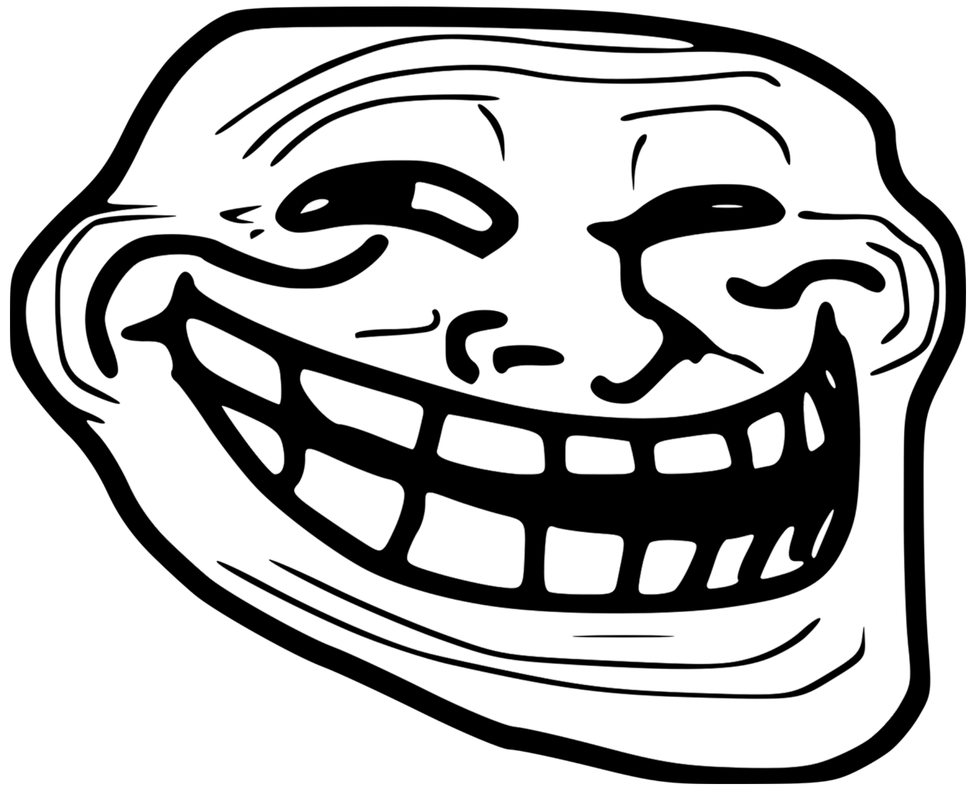
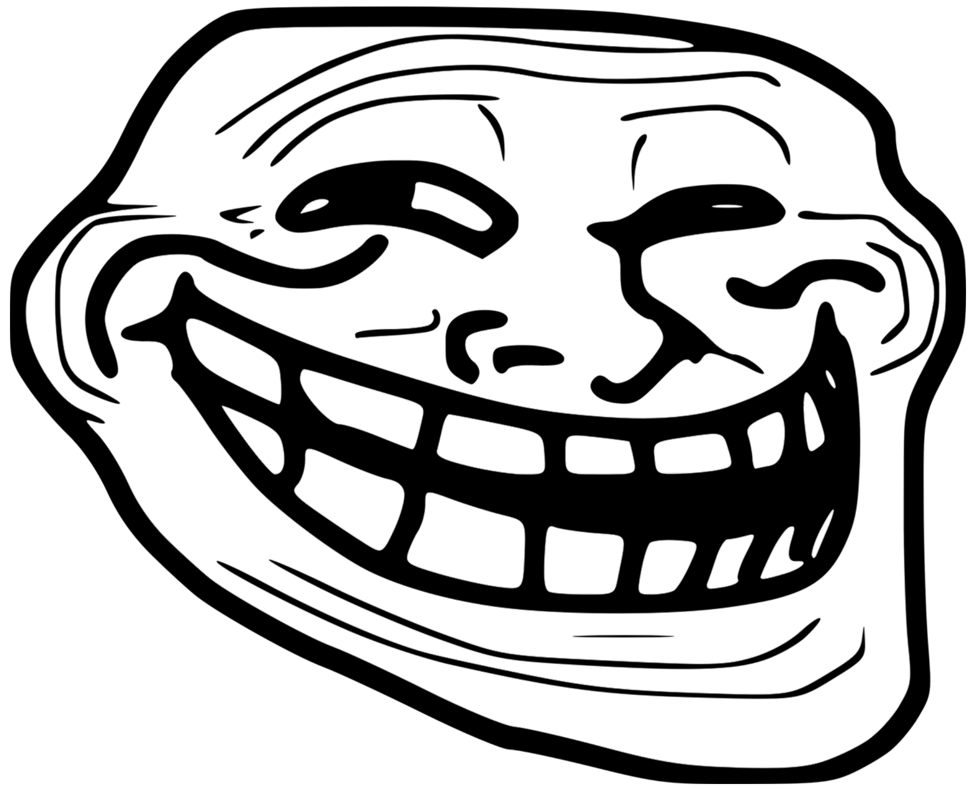
[… REINTERPRETING …]





“The effect of **that** monad is **prepending** to the log.”



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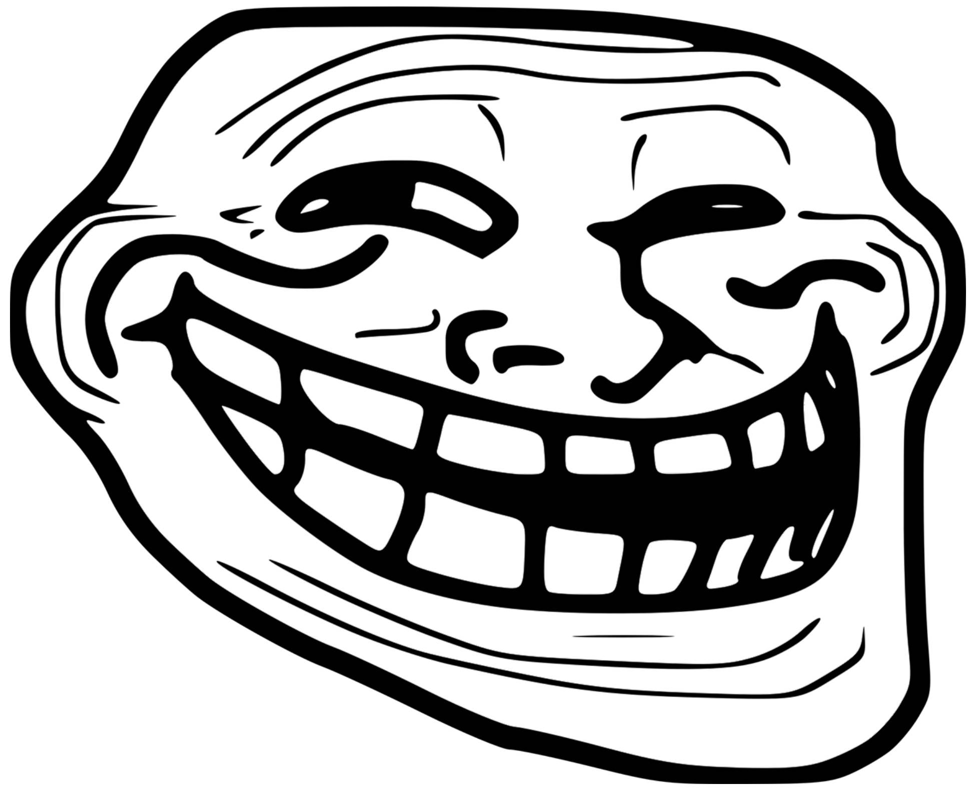

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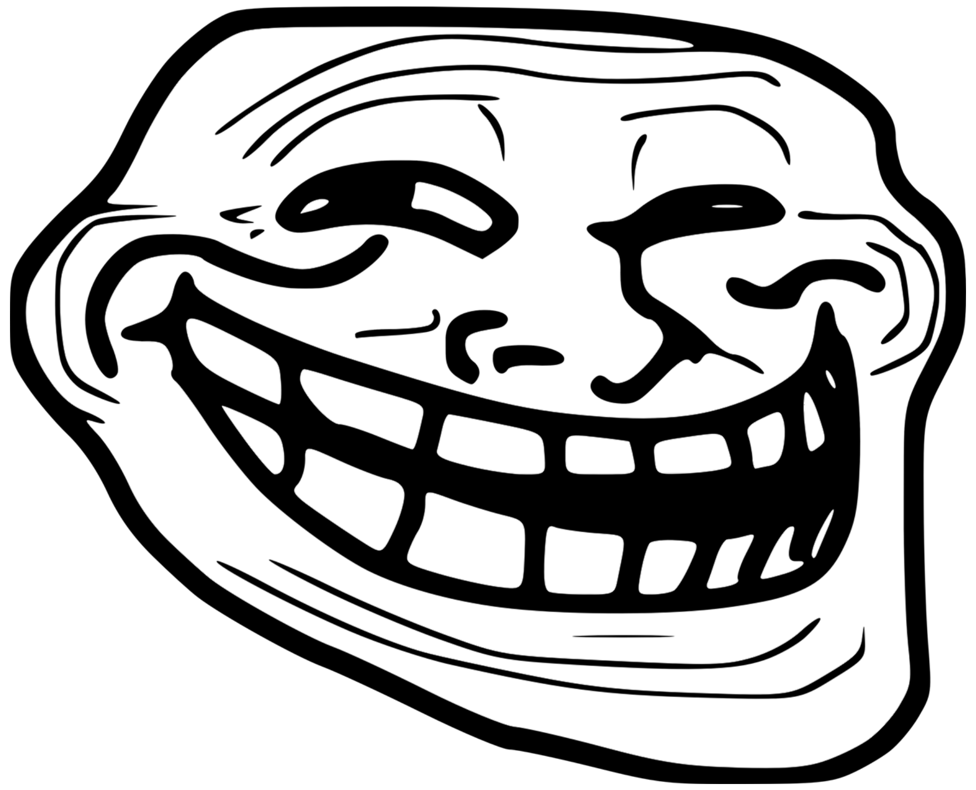

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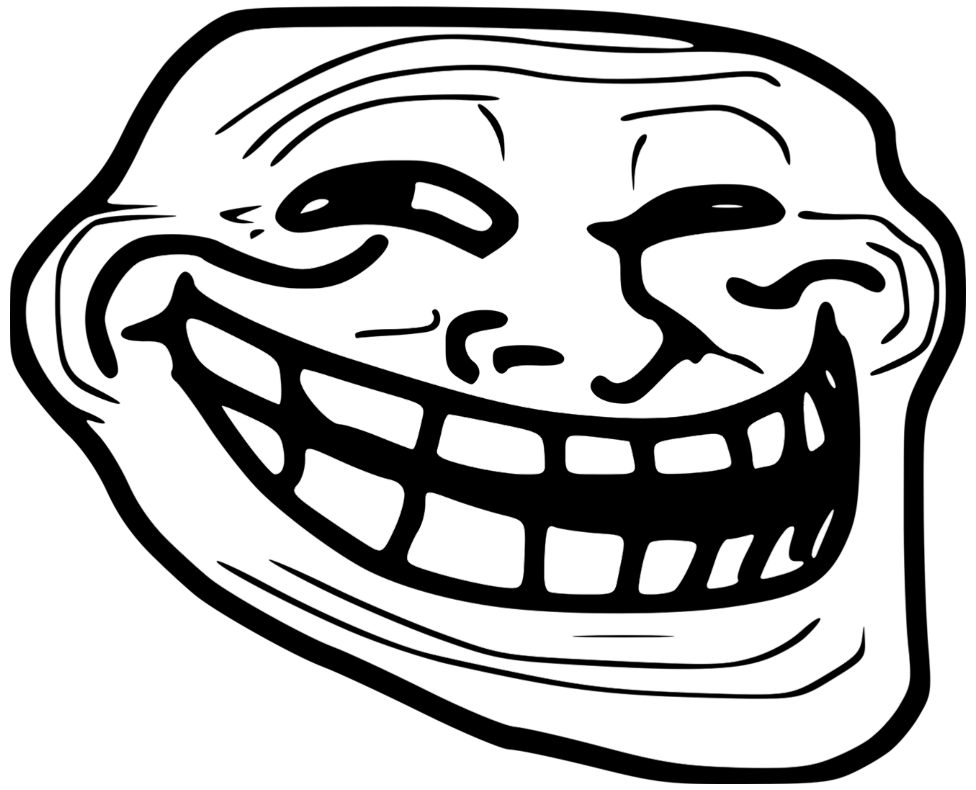

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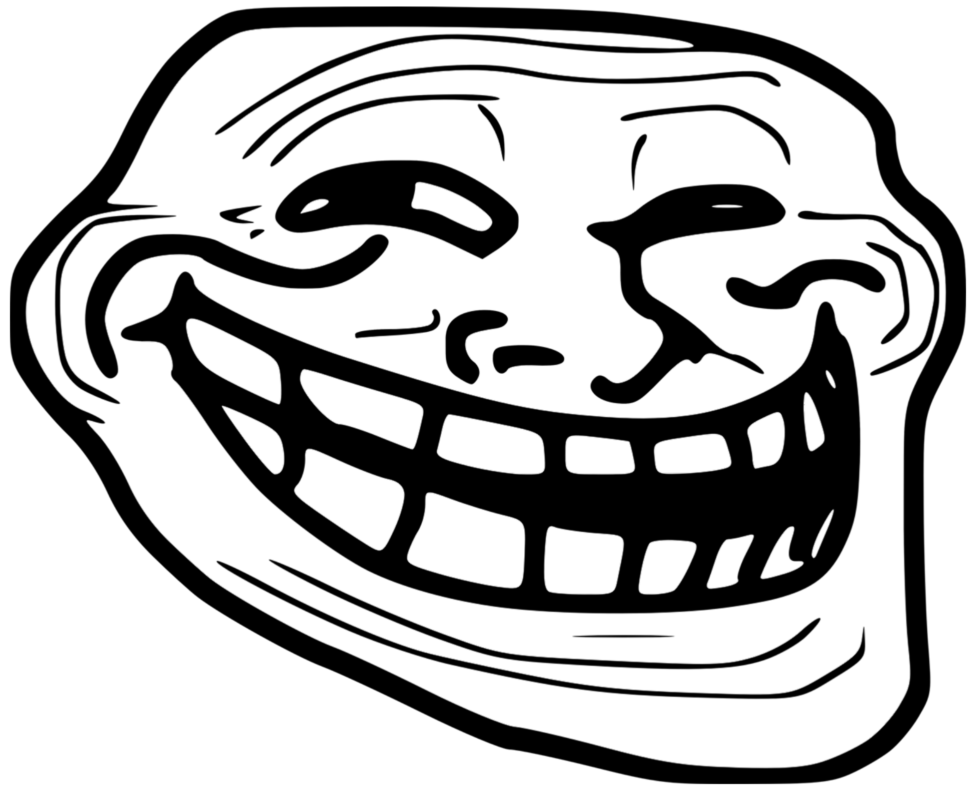

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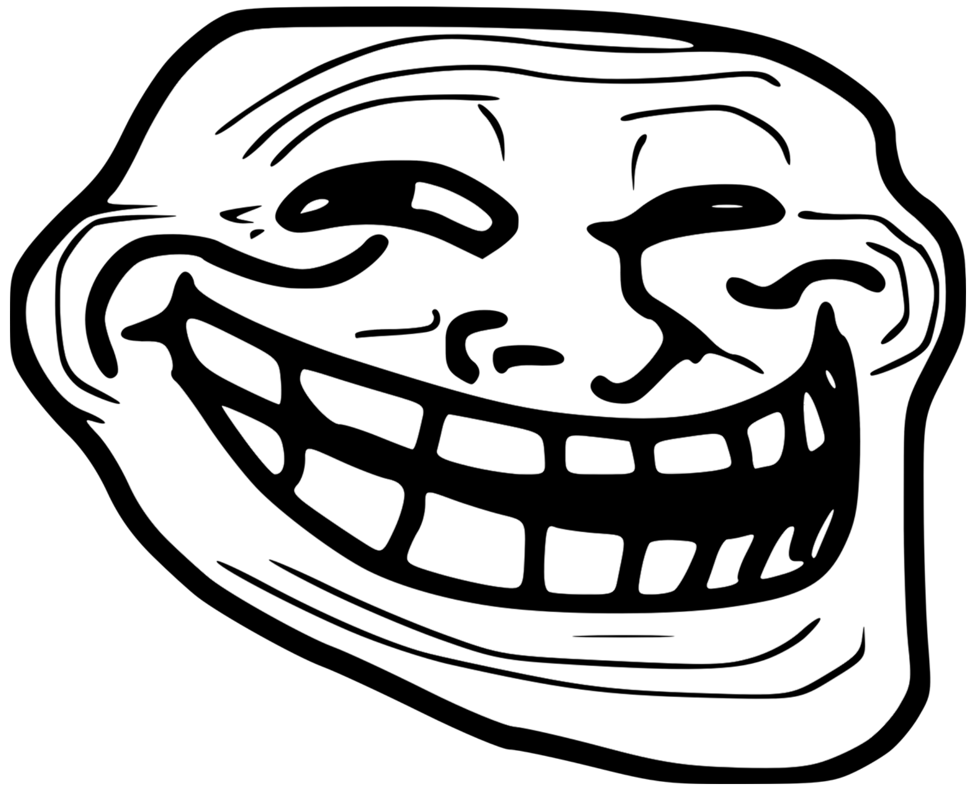

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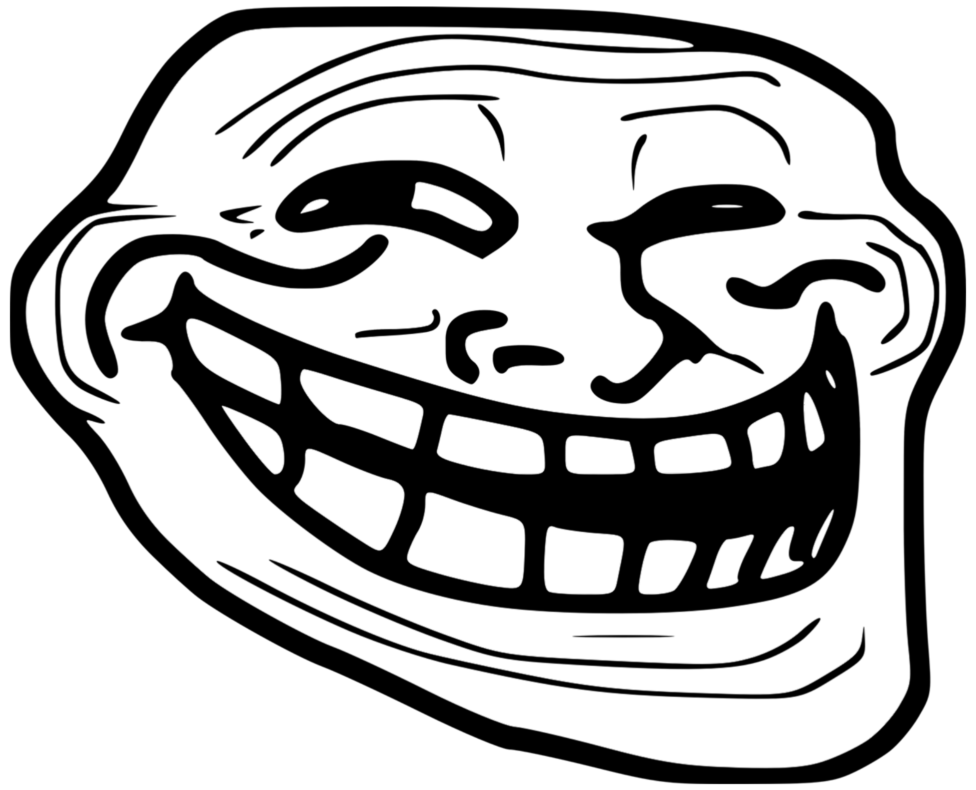

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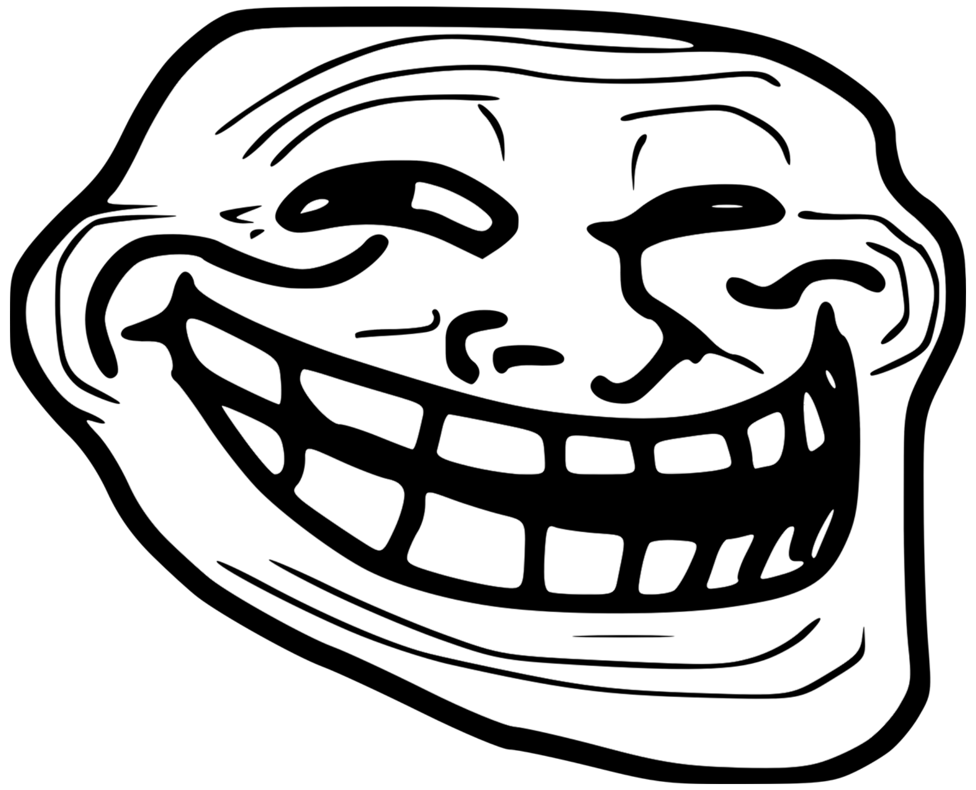

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- Sequential composition ≠ sequential **execution** (e.g. monads in **<=**)
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IT IS SEQUENCING!!!

(the behavior of twisted `Writer`)

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Just say NO! 🙅‍♂️

Don’t feed the trolls.

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Up Next:
Concurrent, Non-deterministic `Writer`

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But first, a quick introduction to

Libretto,

a *concurrent-by-default* DSL embedded in Scala.

Libretto for Scala Programmers

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`A => B`	`A -○ B`	Functions in Libretto are *linear.*
`Promise[A]`	`-[A]`	Cannot be ignored. Cannot be completed twice.

`List` in Scala vs. Libretto

Scala

```
type List[A] = Either[Unit, (A, List[A])]
```

Libretto

```
type List[A] = One ⊕ (A ⊗ List[A])
```

Libretto `List` : Intuition

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`List[A]`

Libretto `List` : Intuition

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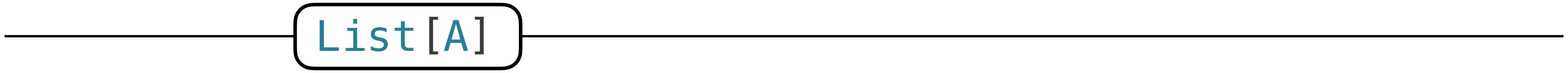

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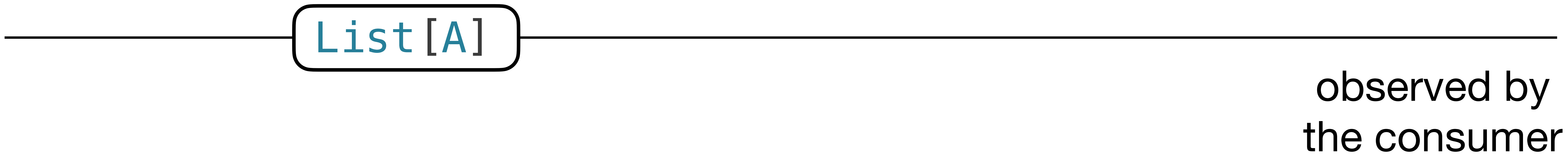

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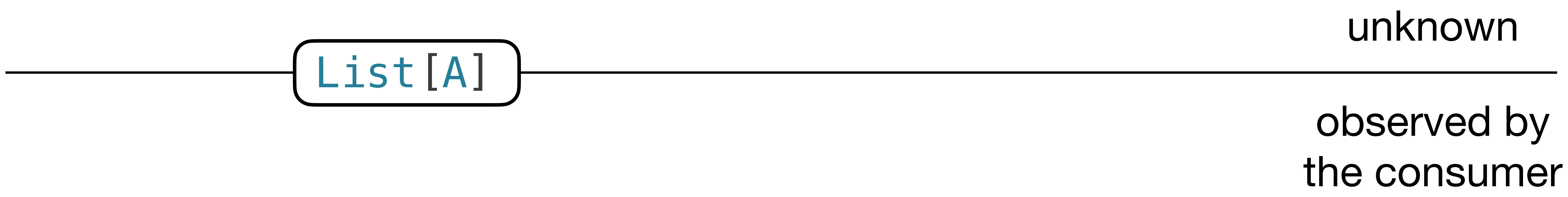

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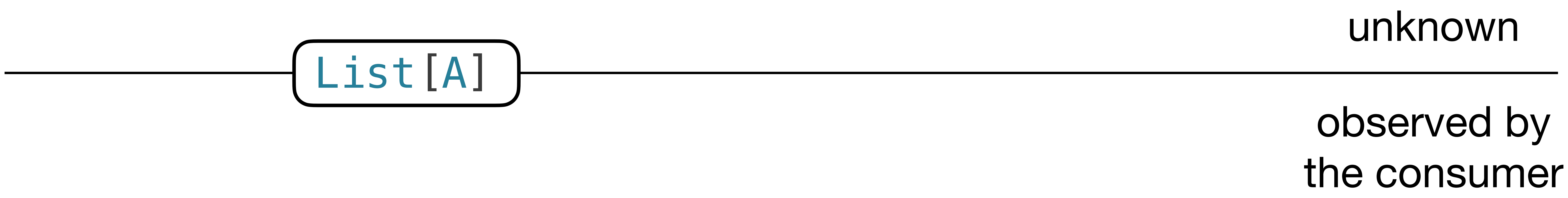

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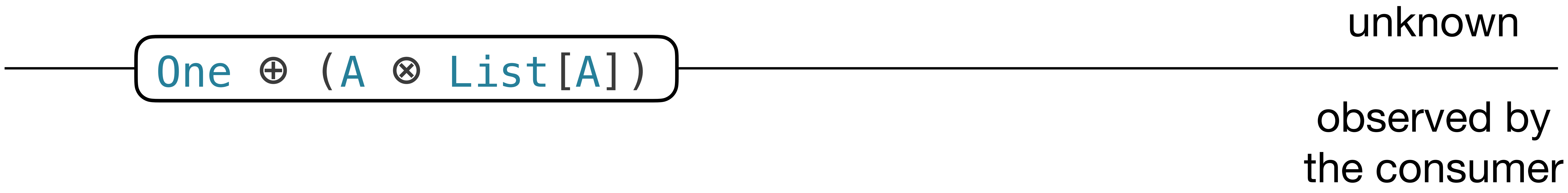

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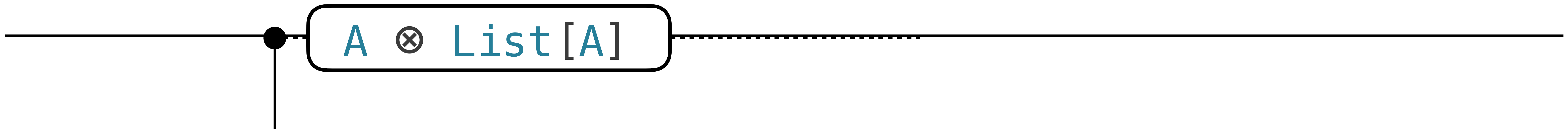

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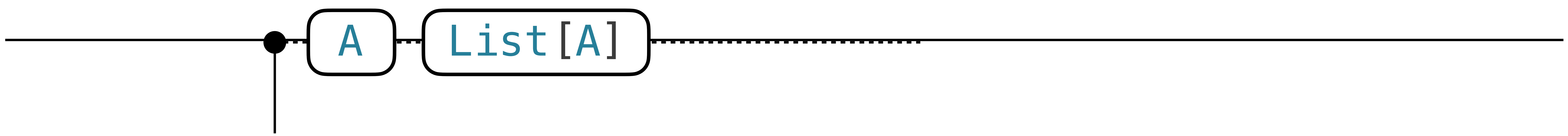

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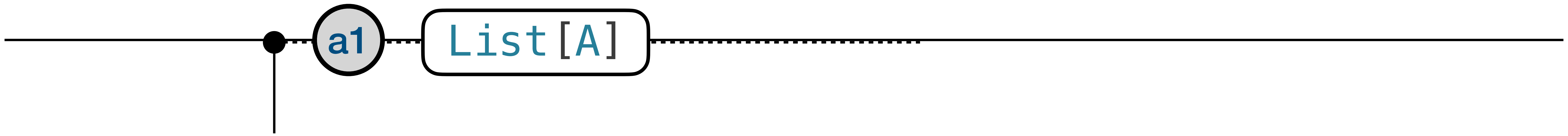

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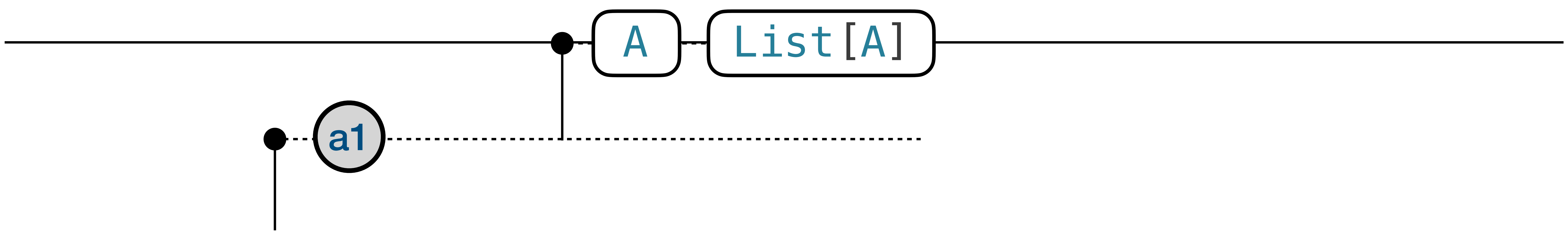

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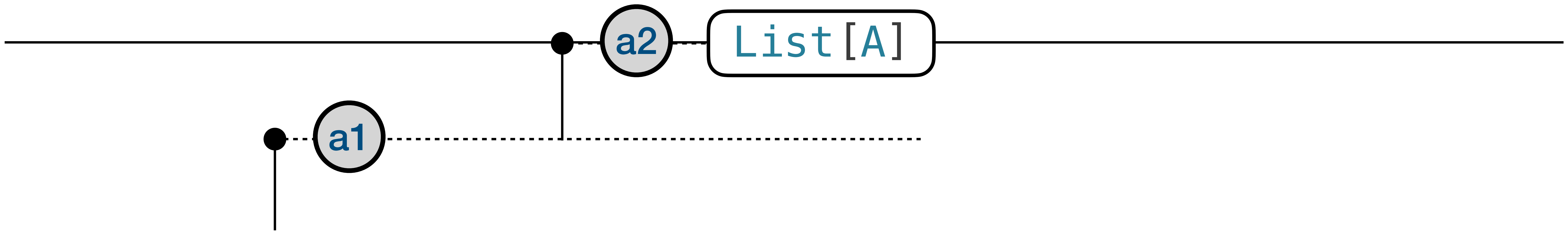

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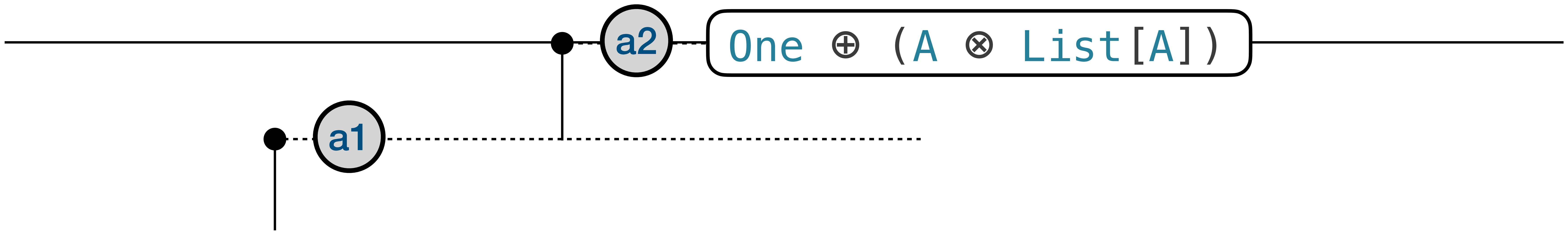

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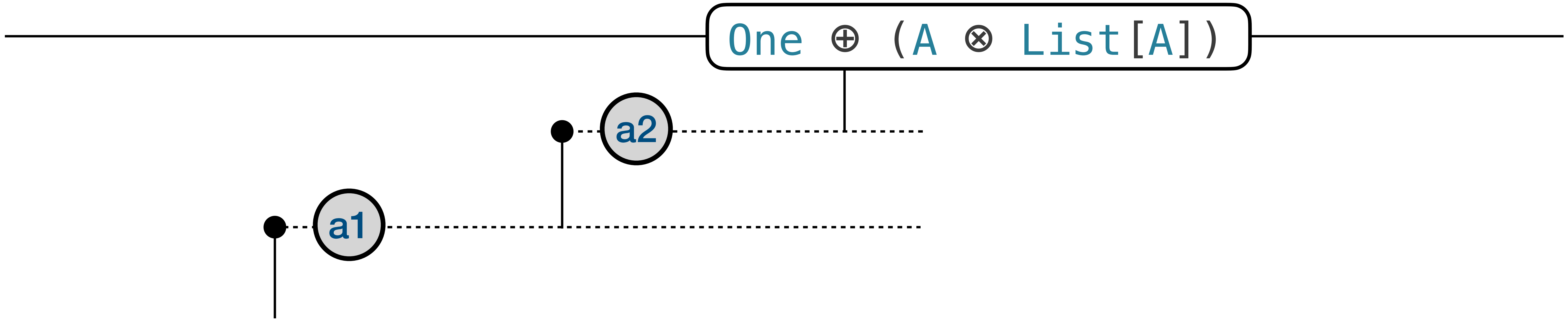

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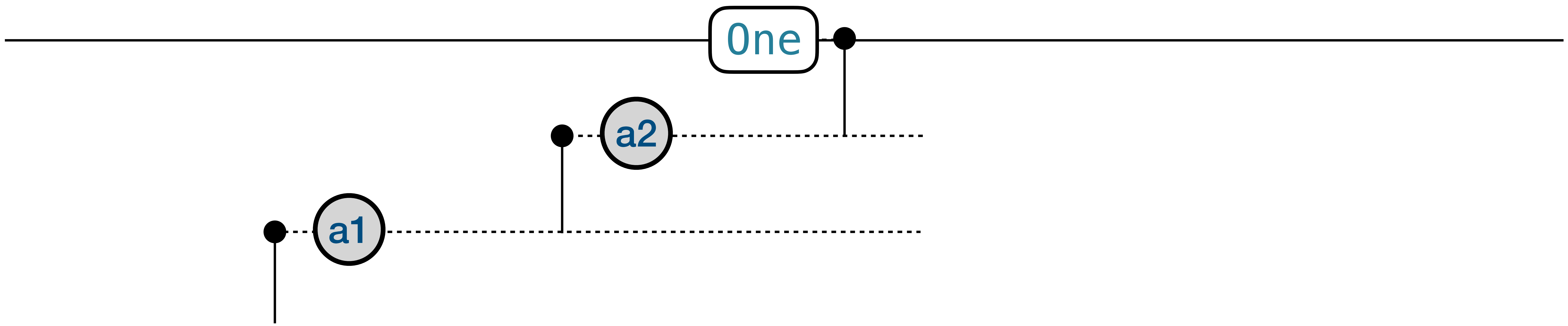

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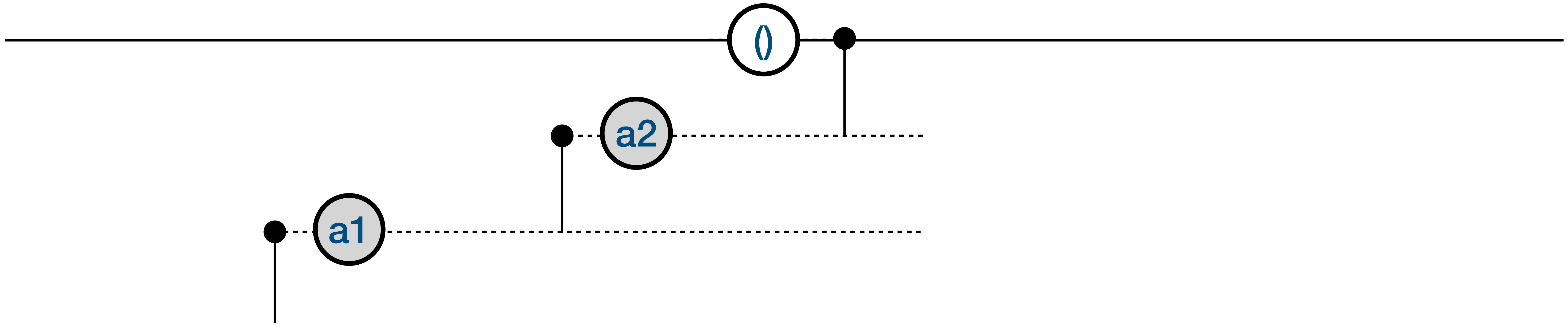

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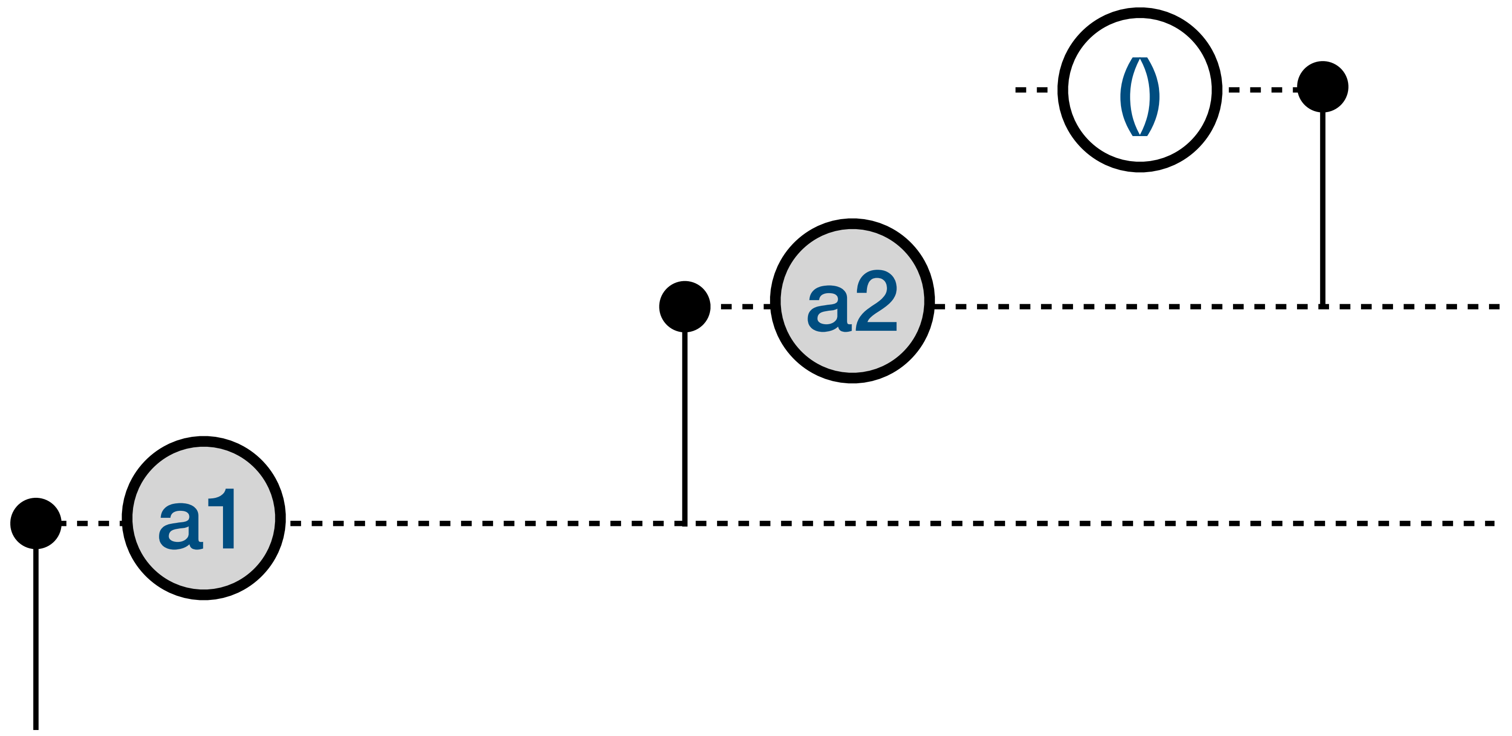

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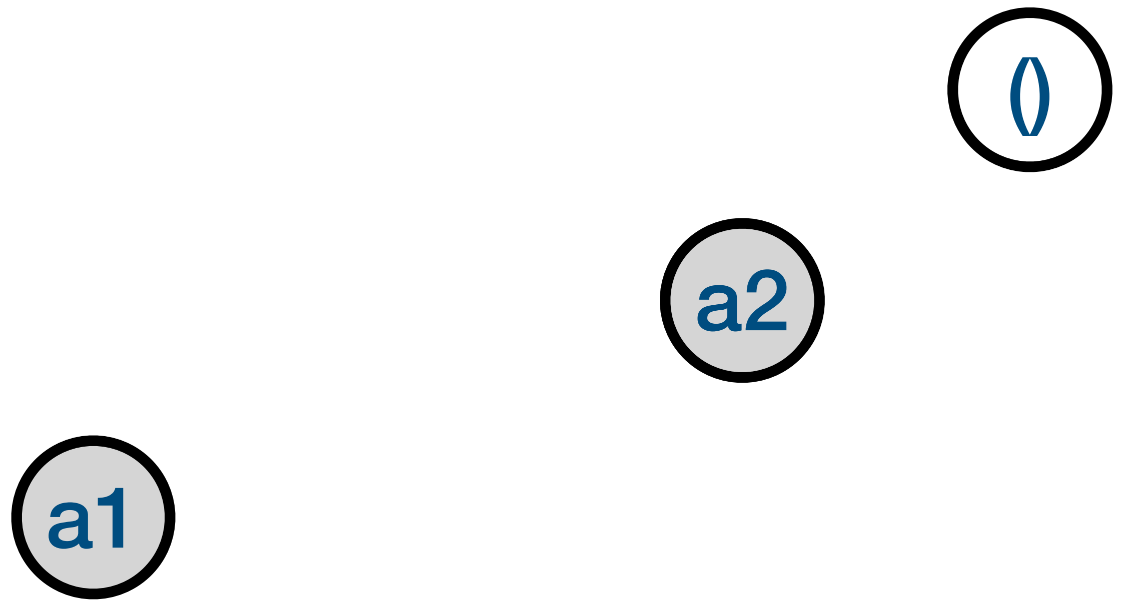
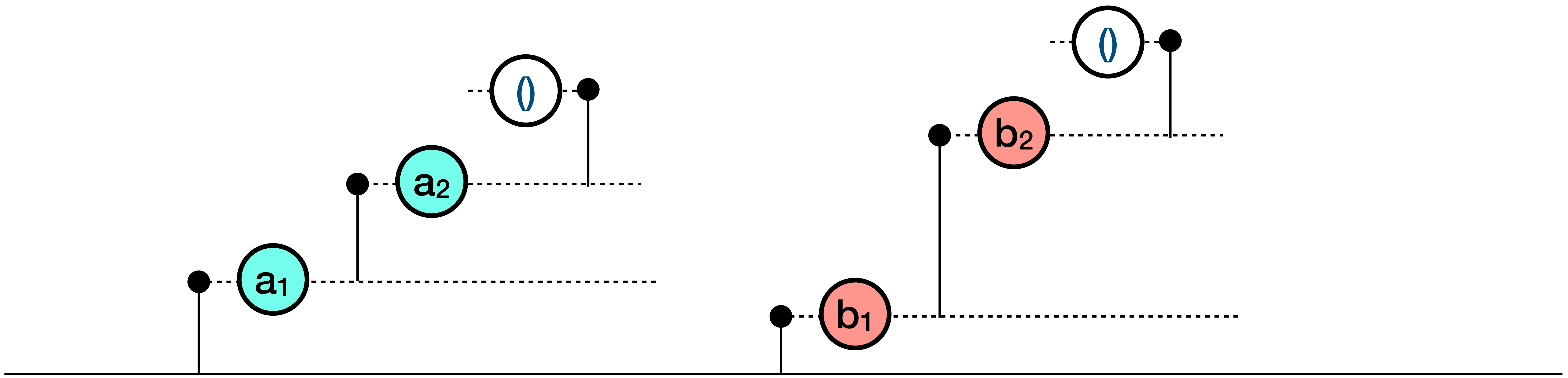


Libretto `List:concat`



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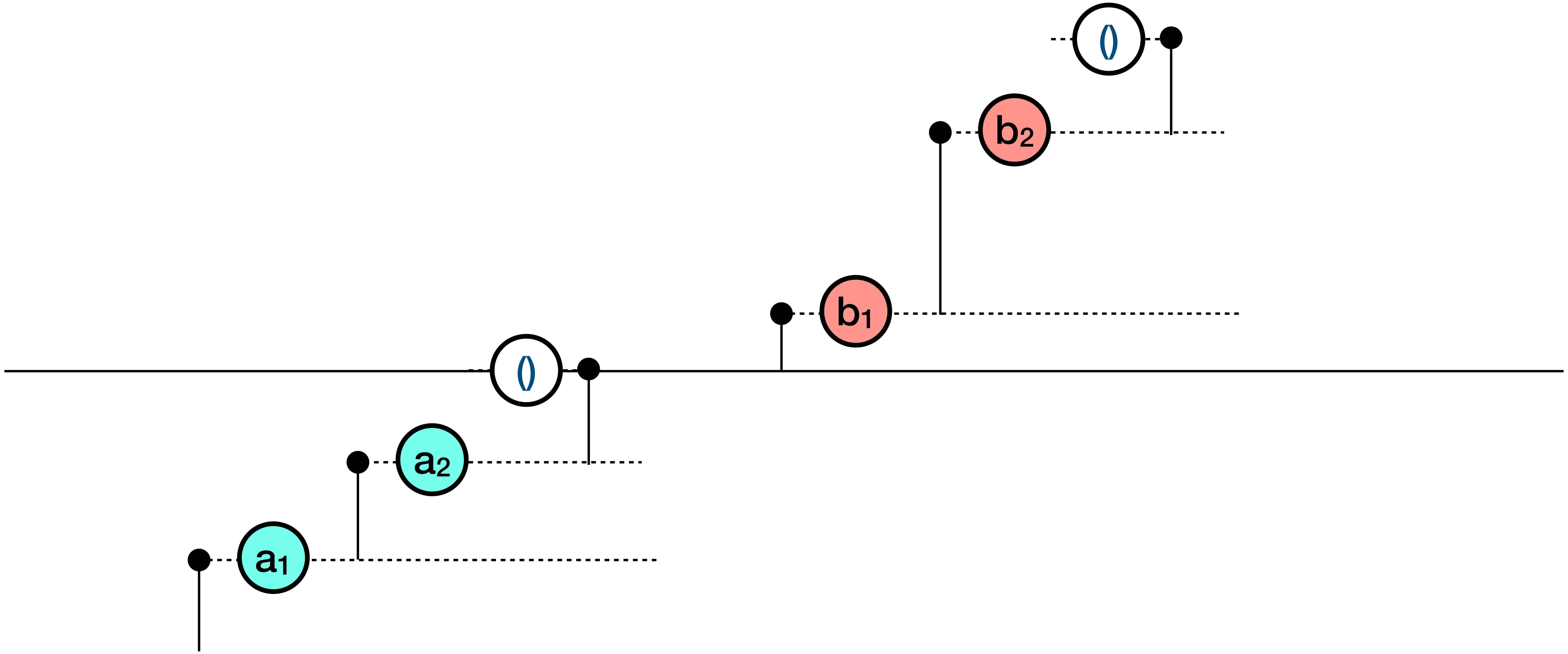

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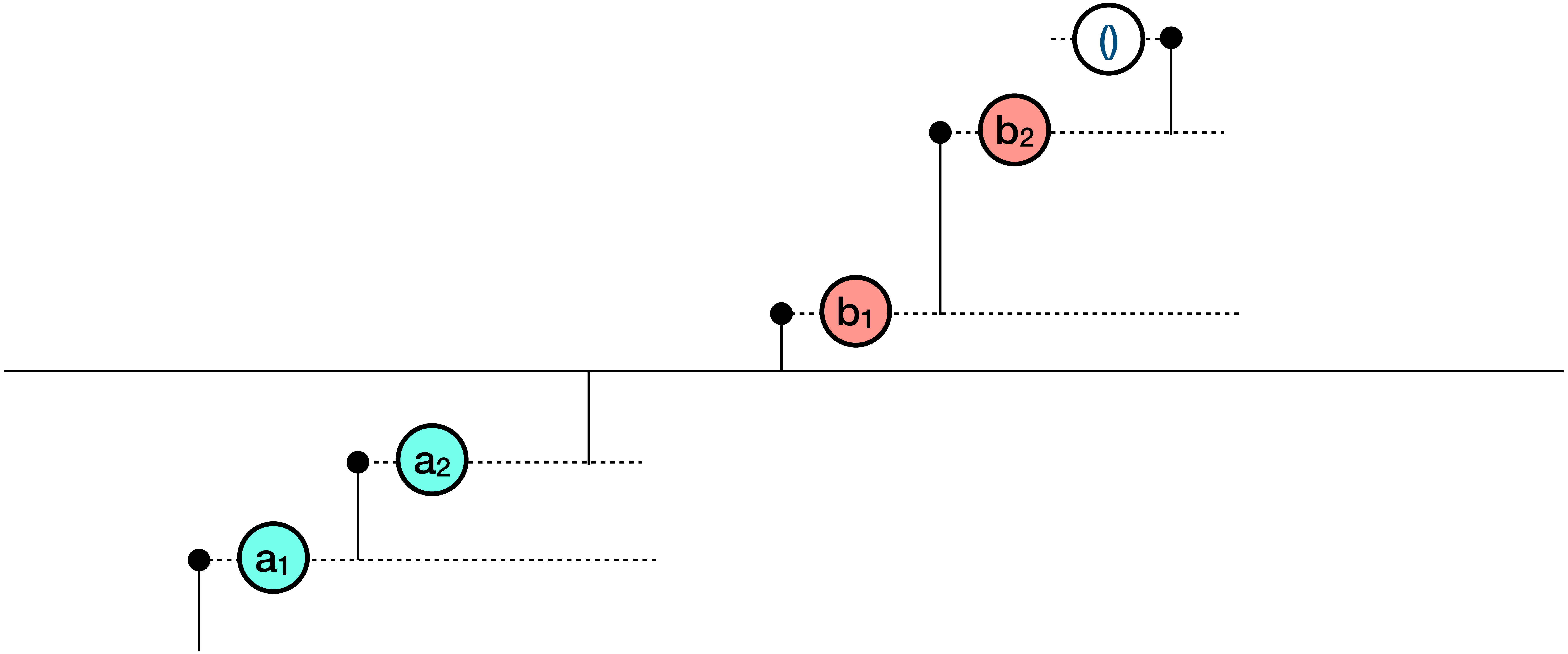

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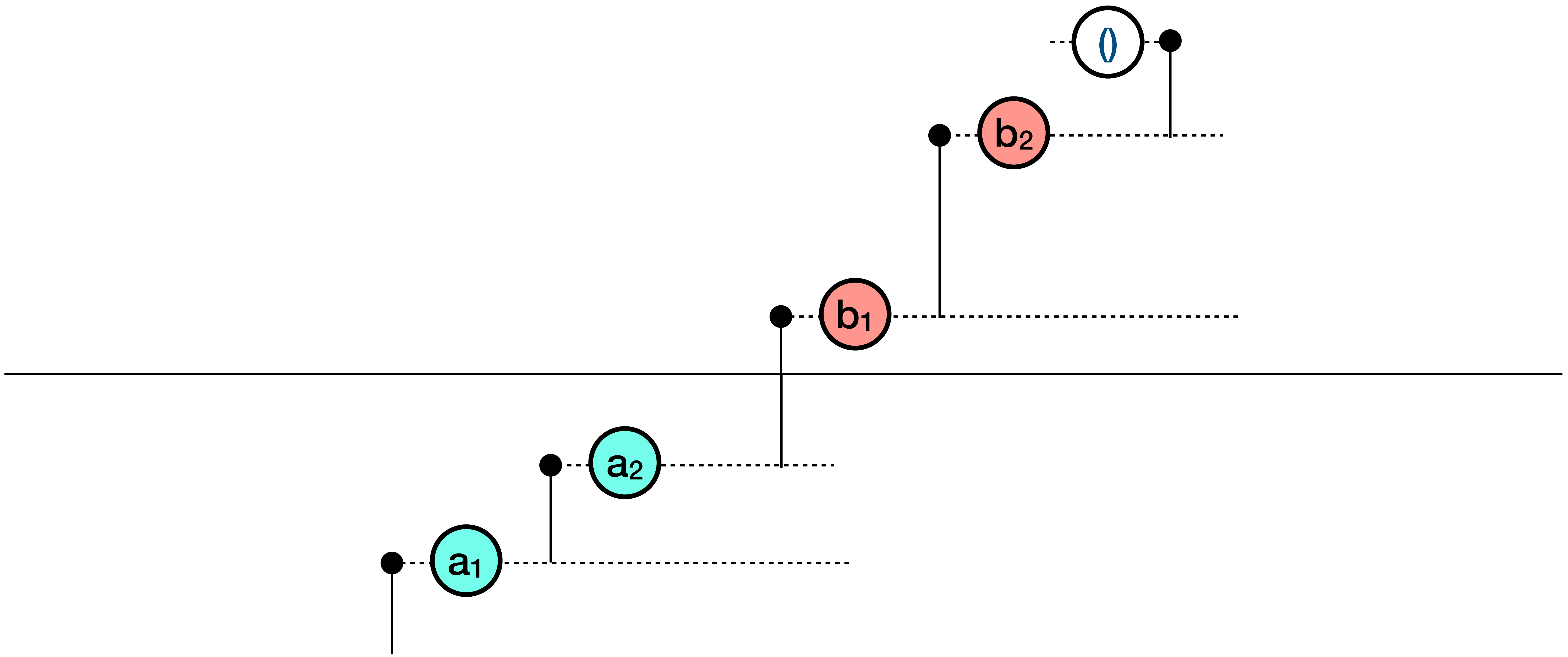

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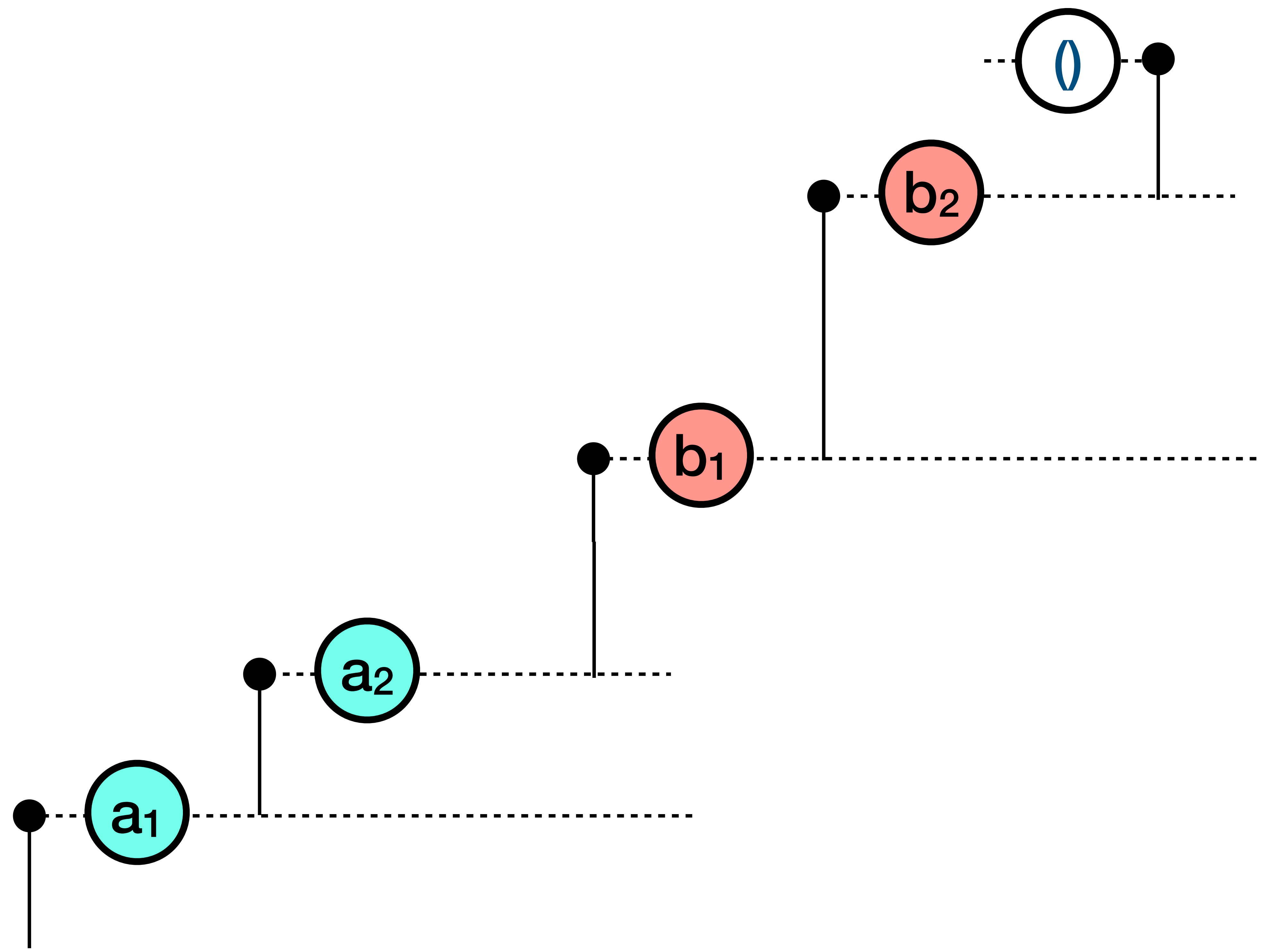

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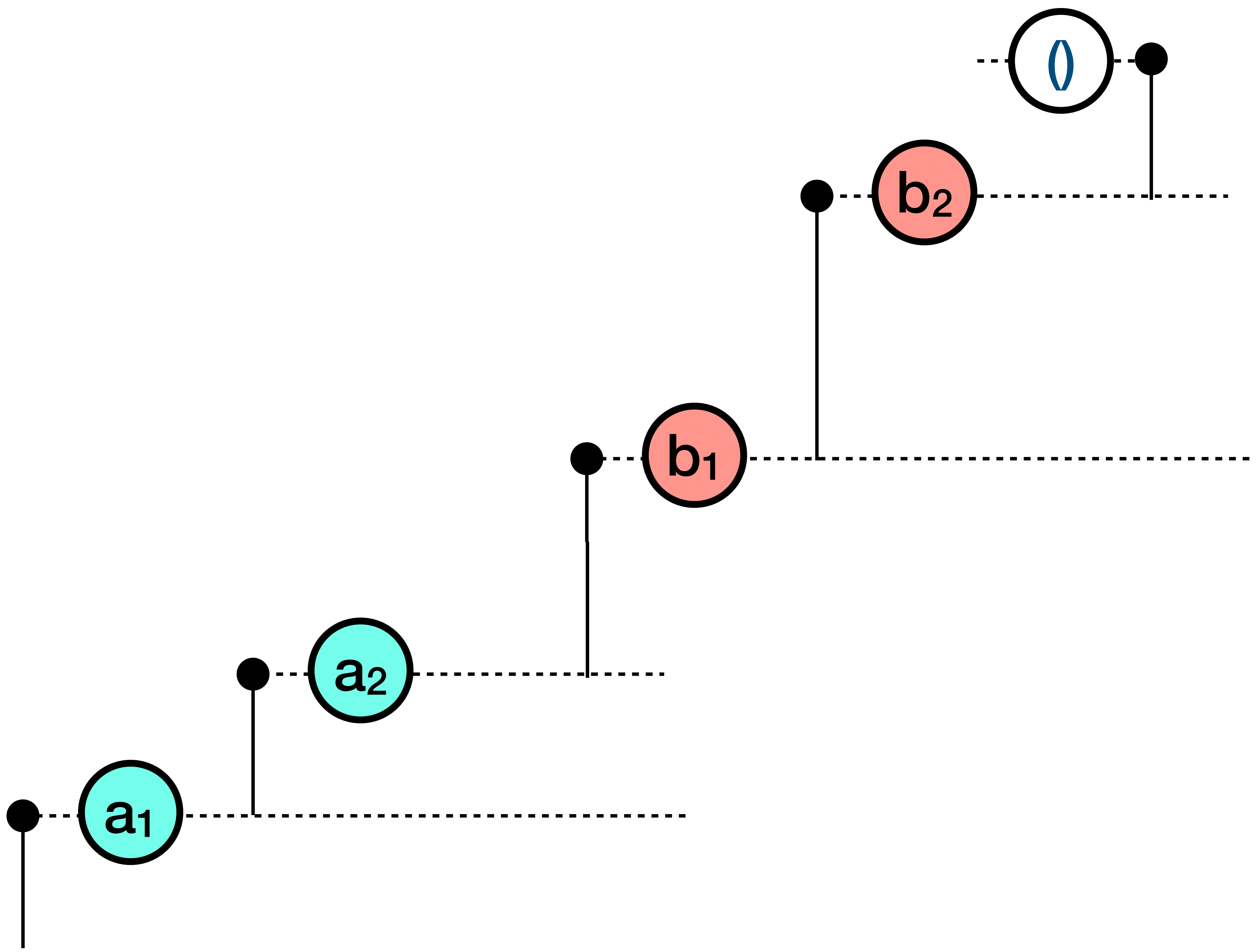
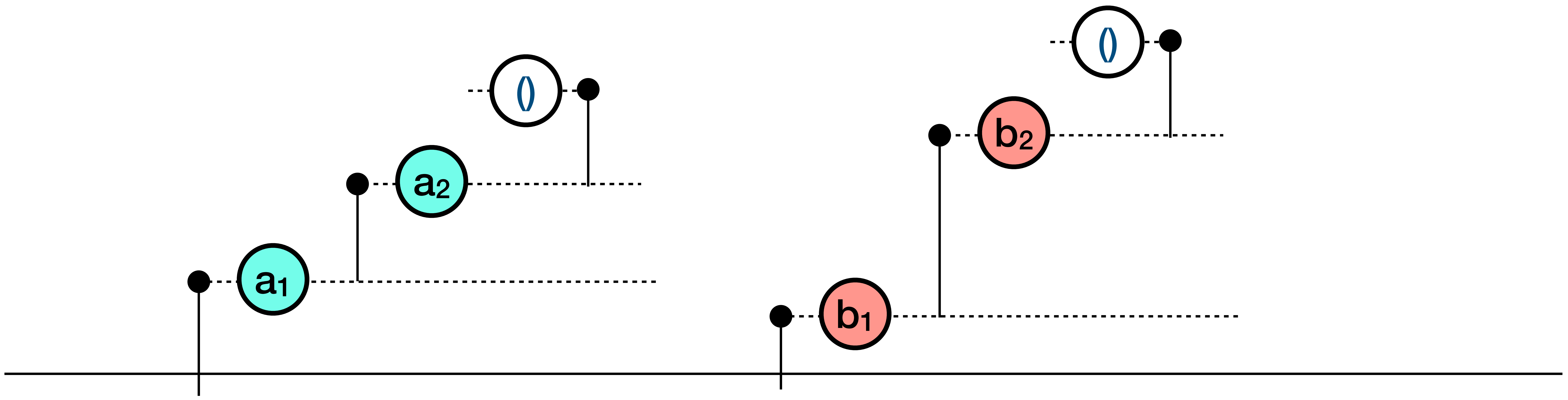


Forms the usual Monoid on `List`

Libretto `List:merge`



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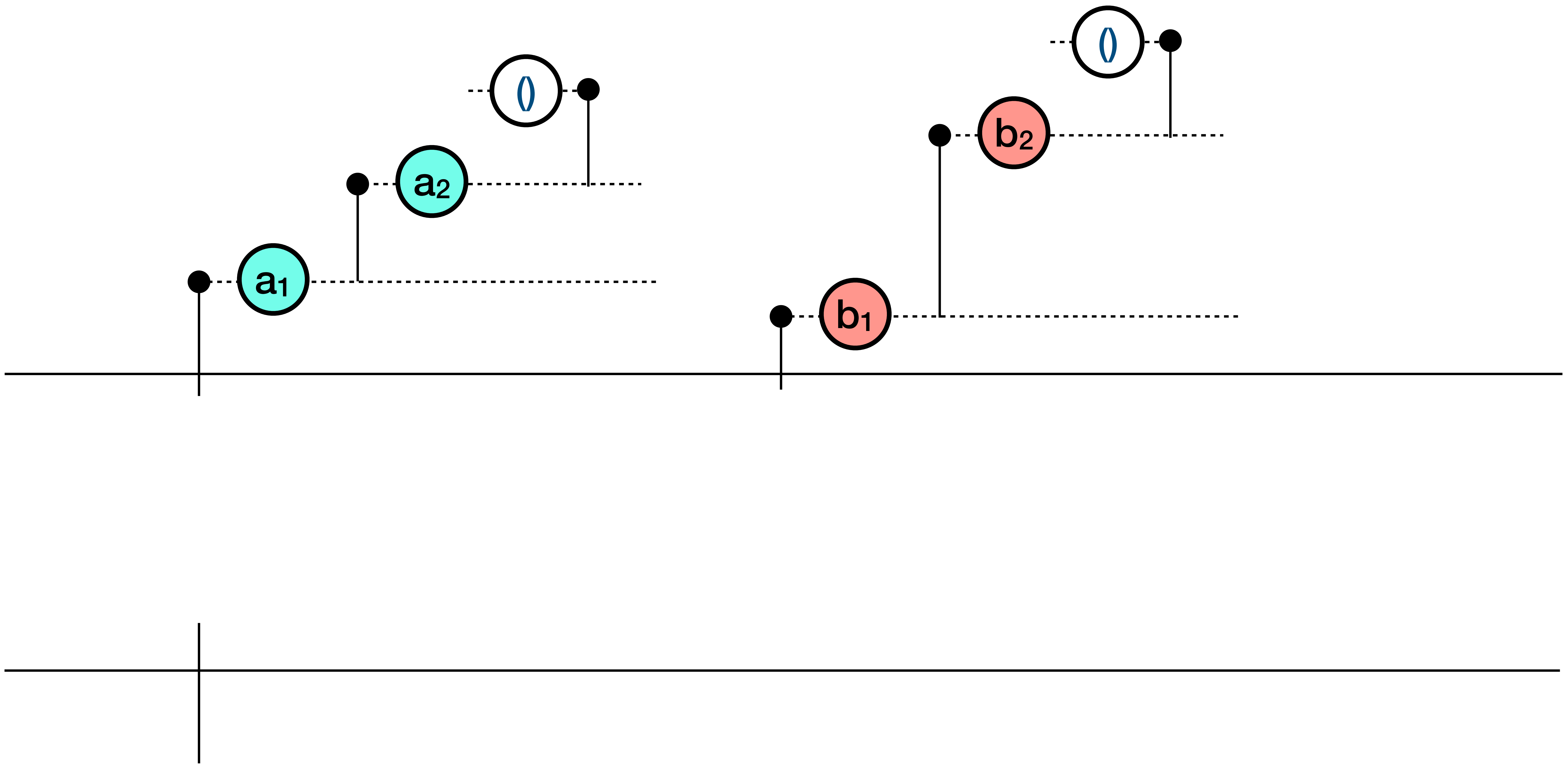

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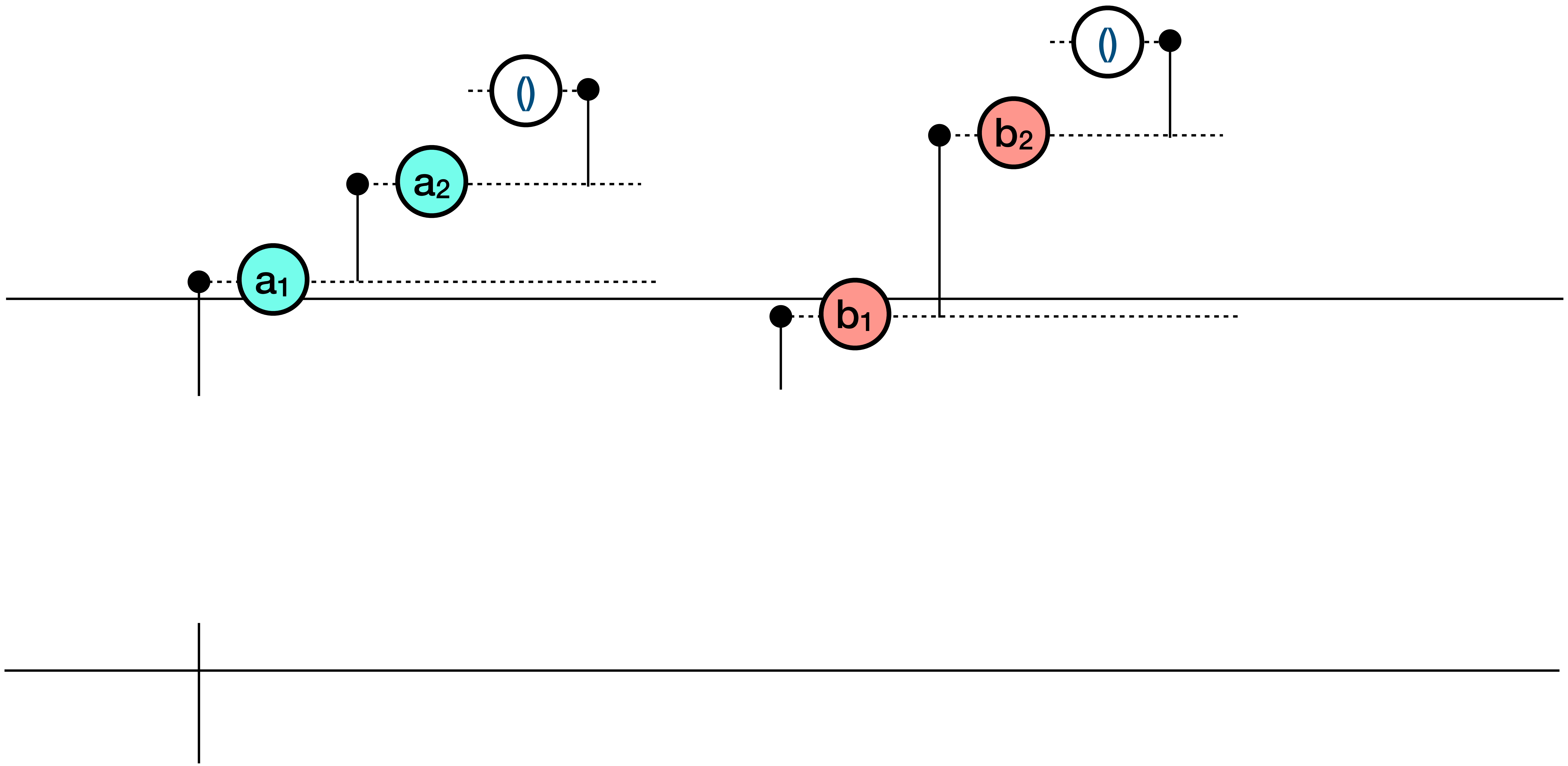

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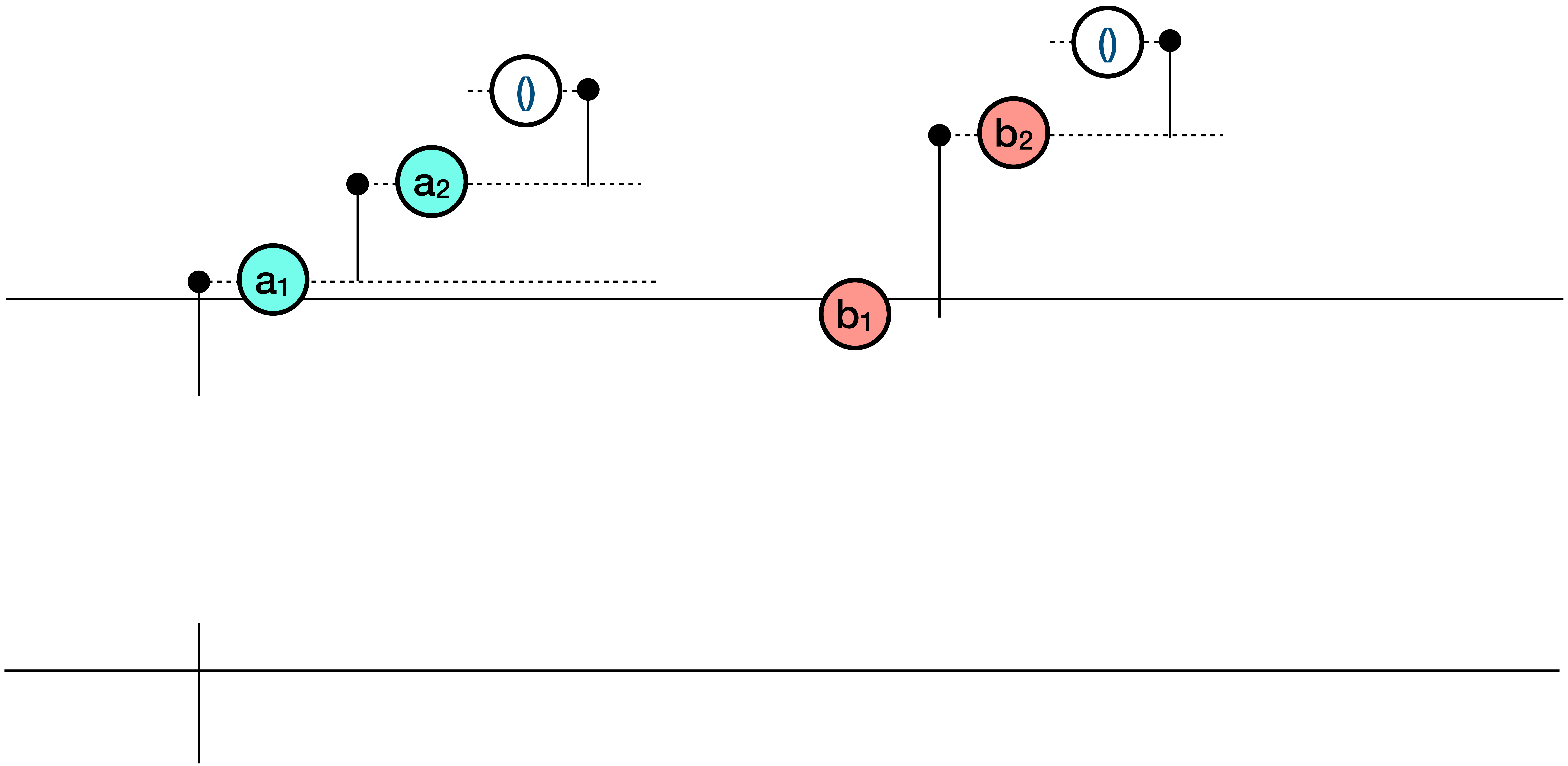

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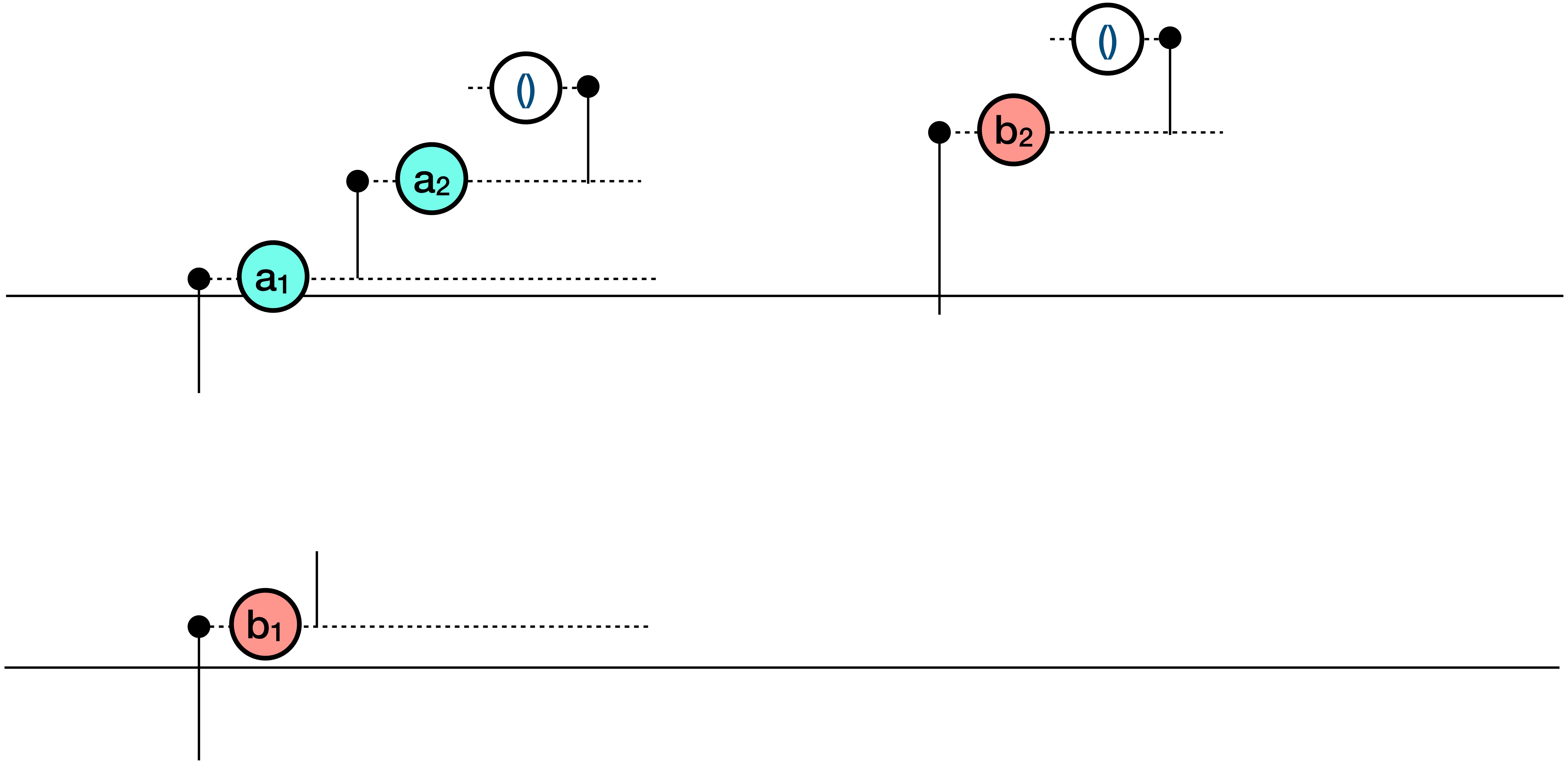

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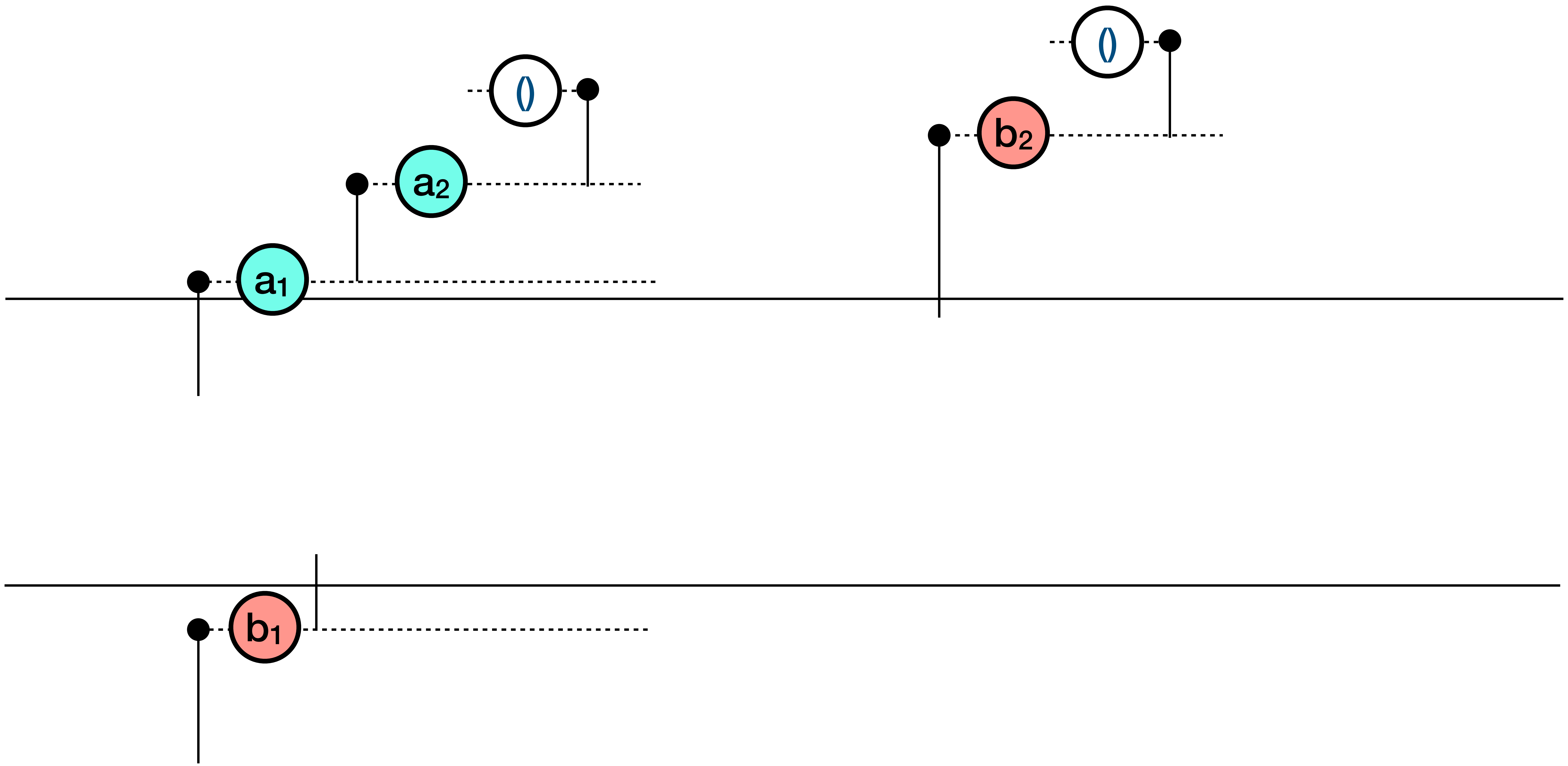

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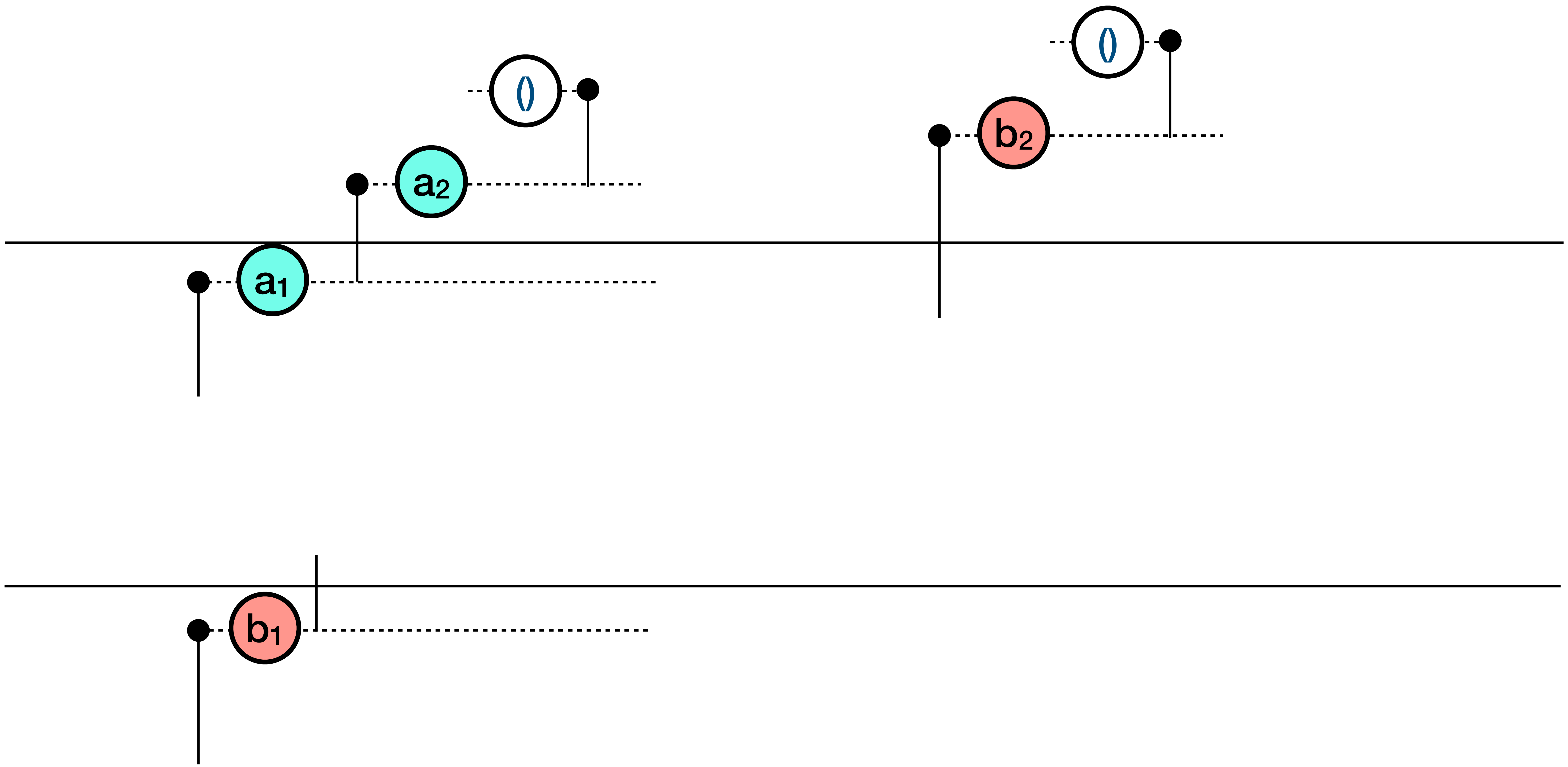

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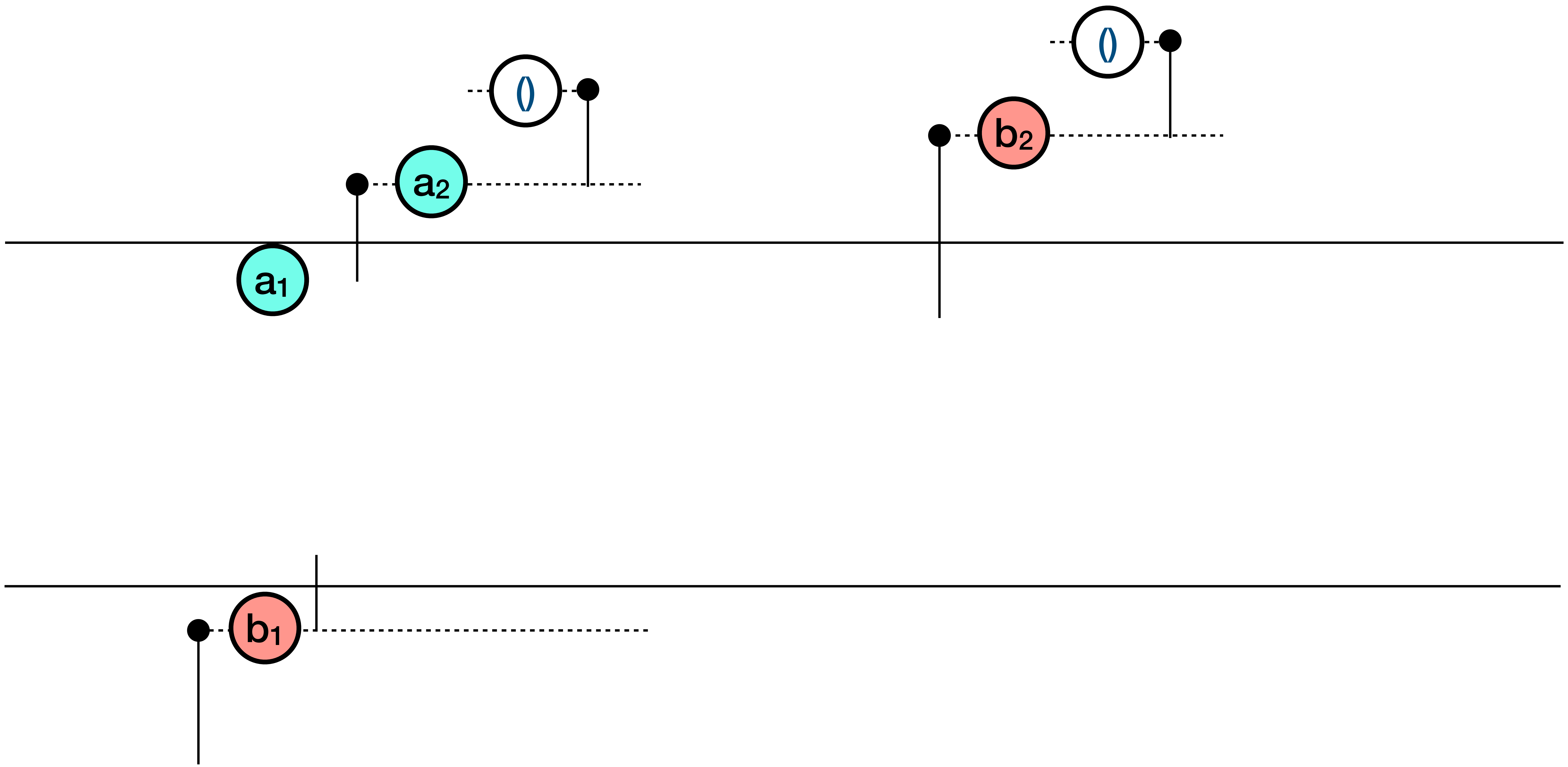

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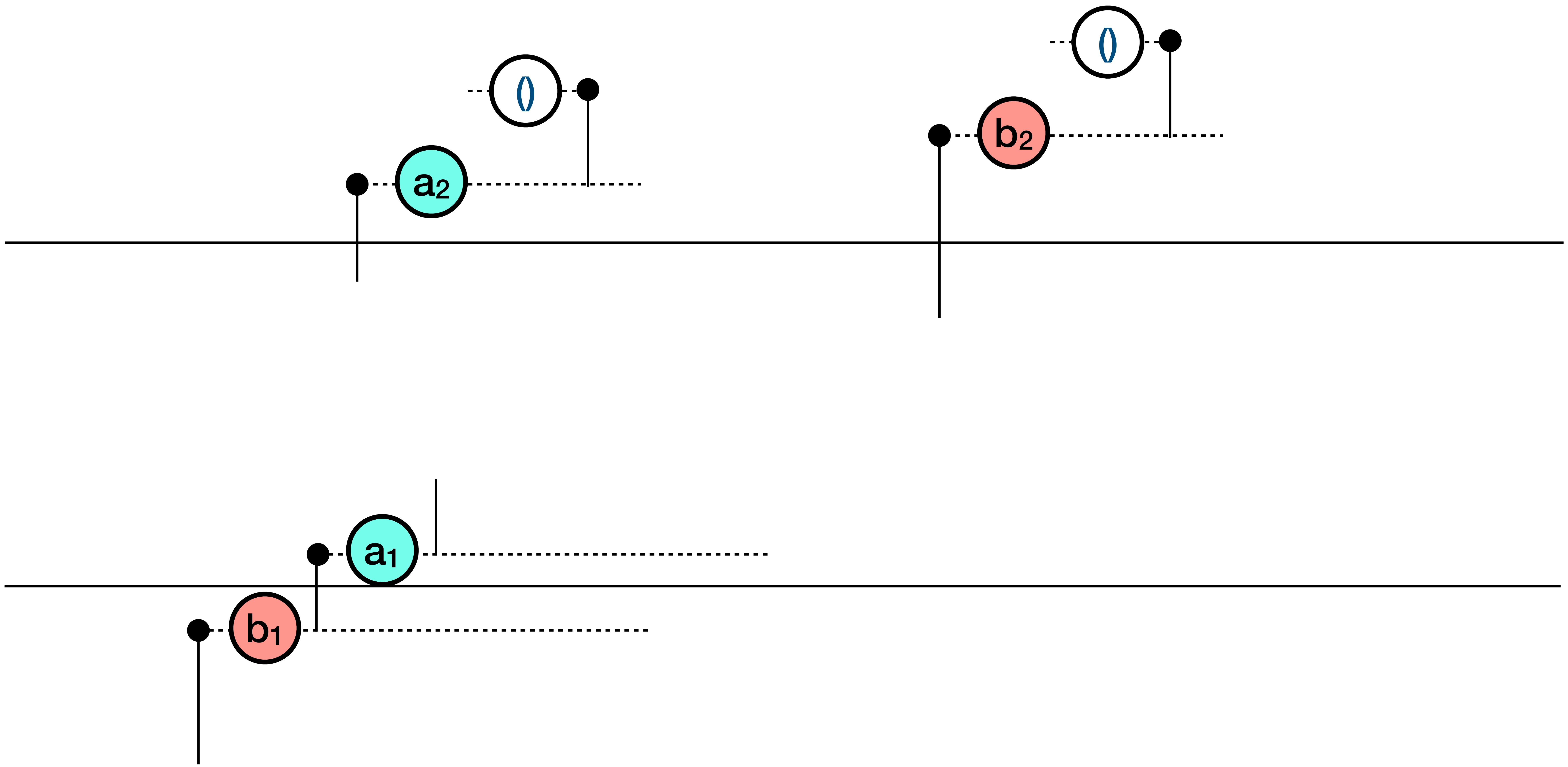

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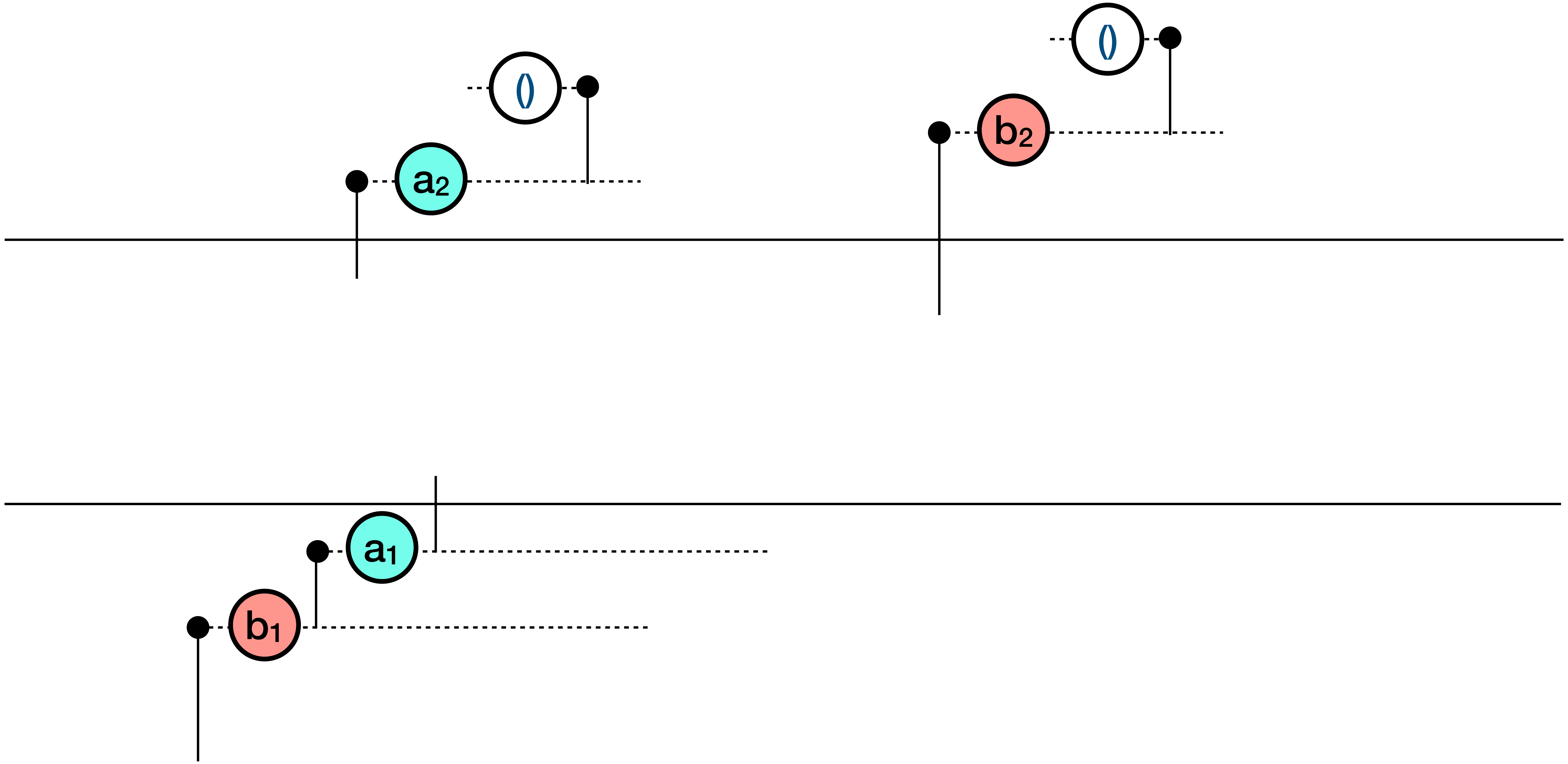

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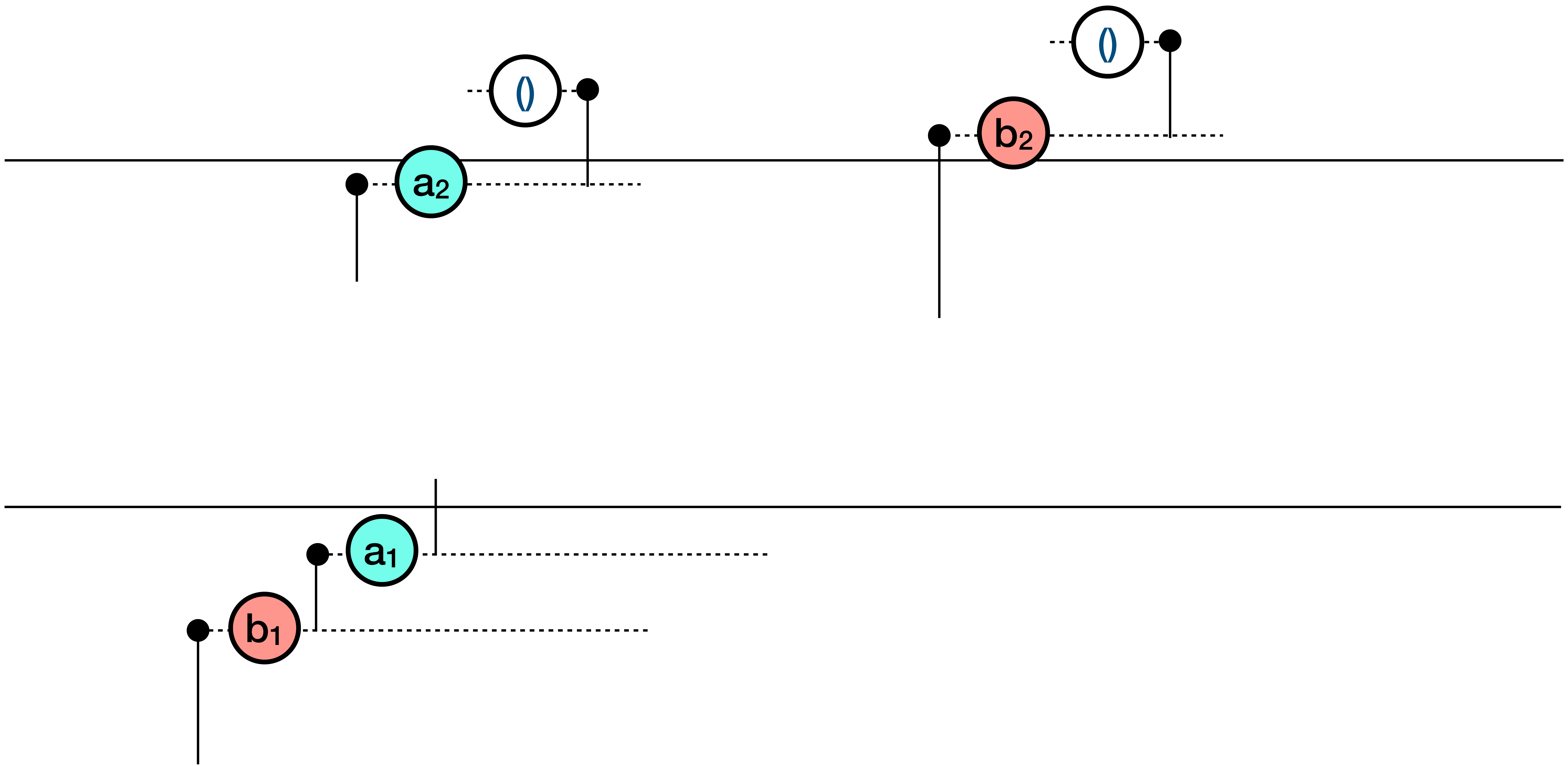

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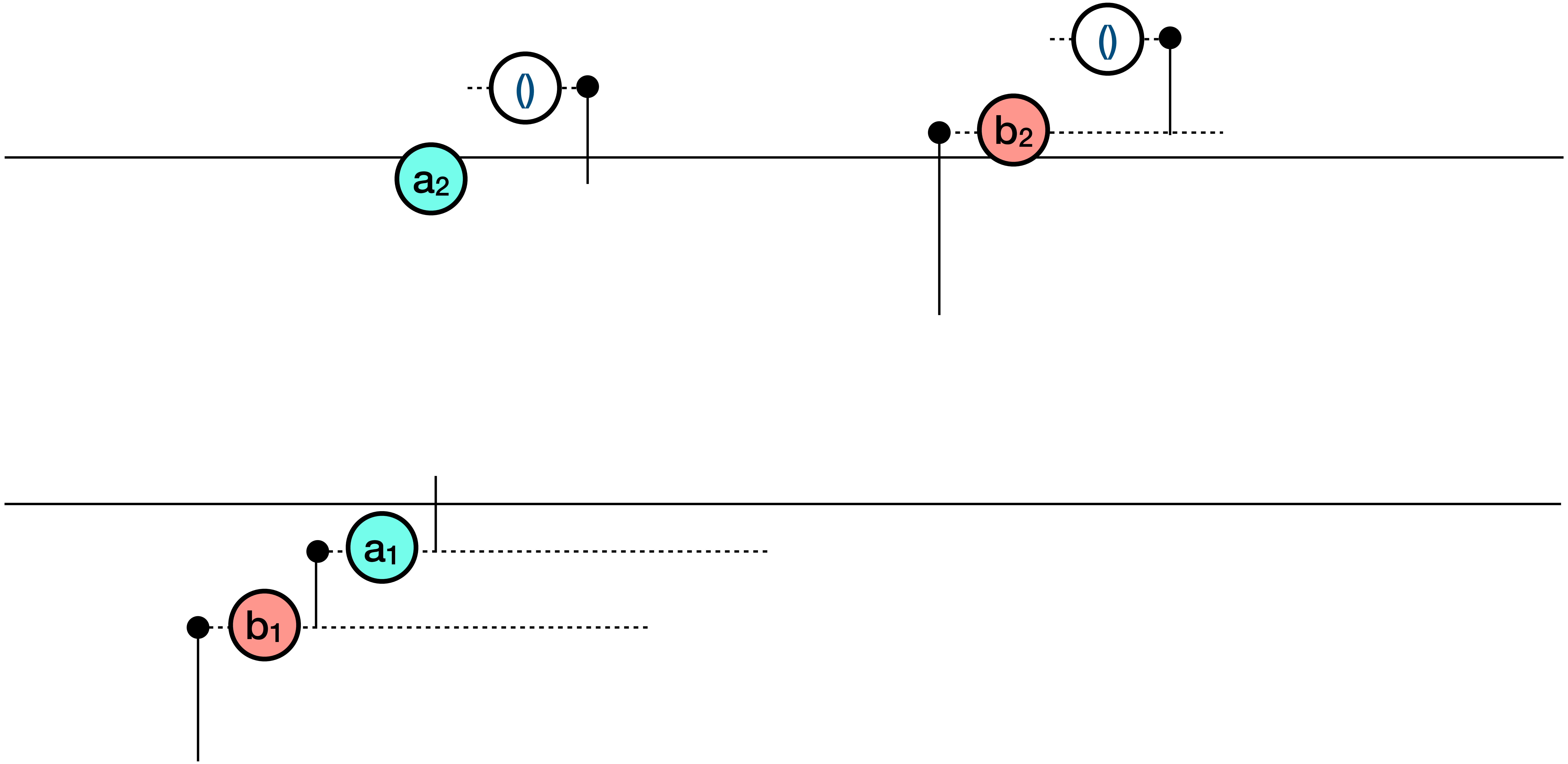

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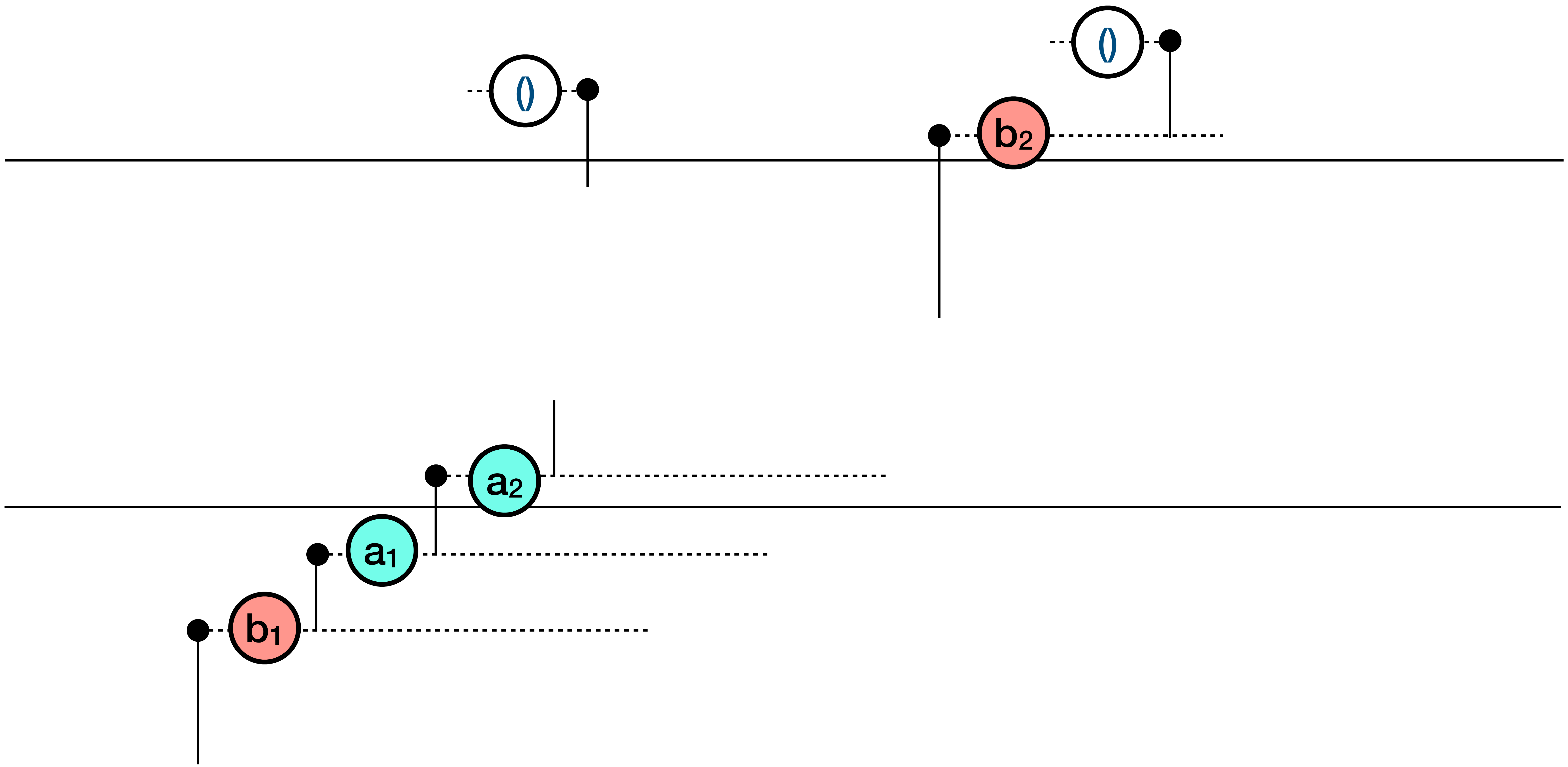

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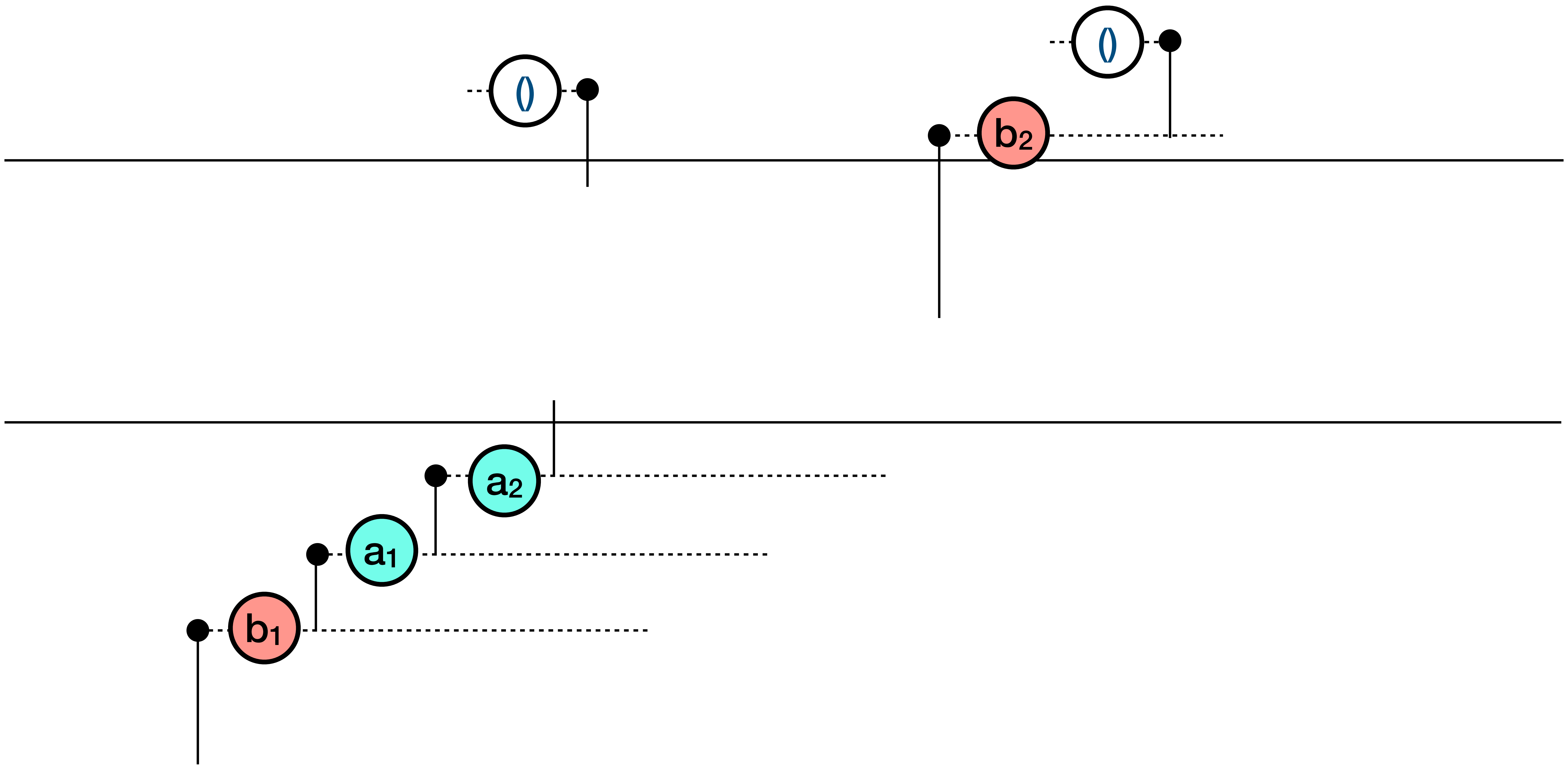

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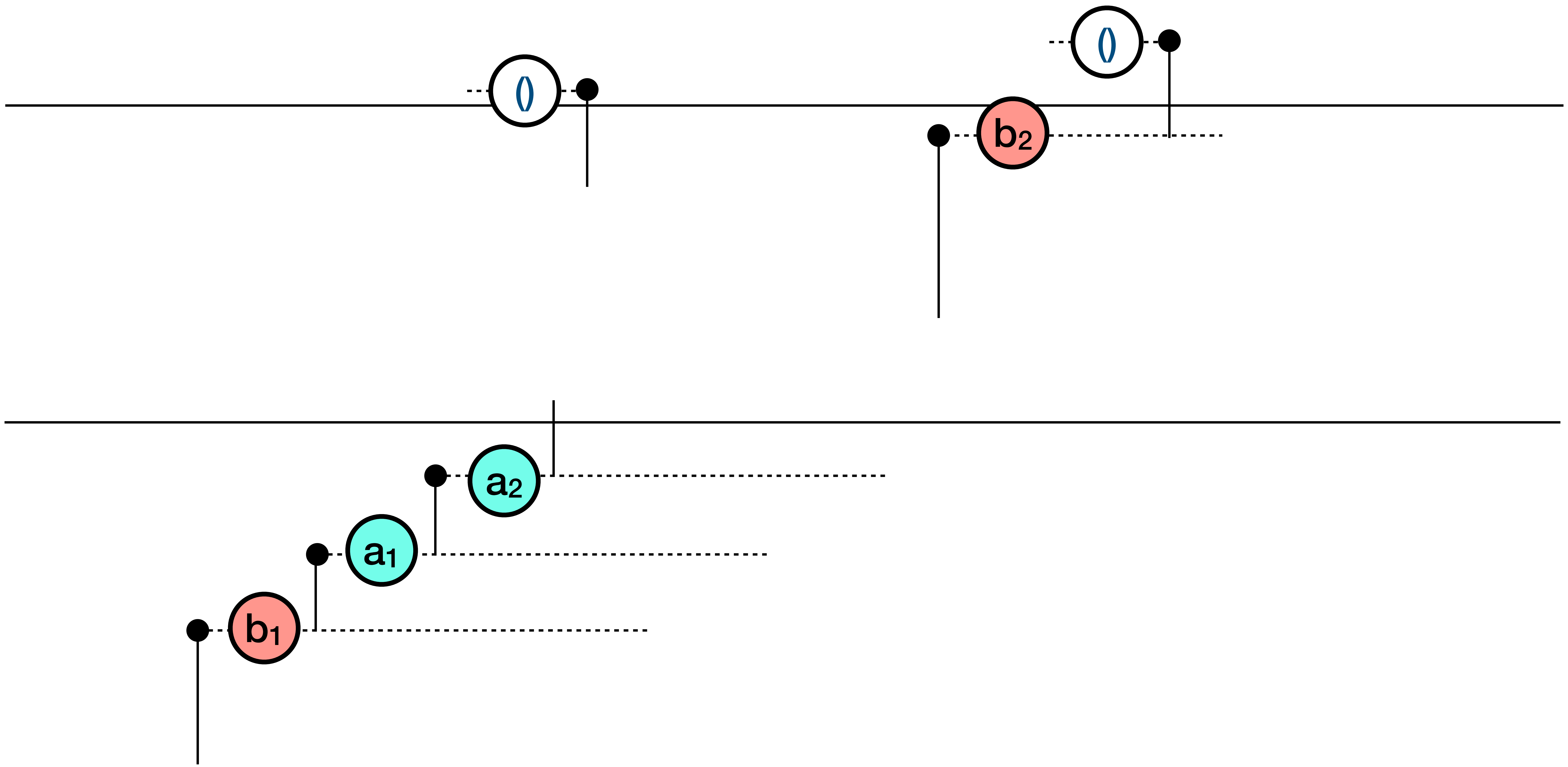

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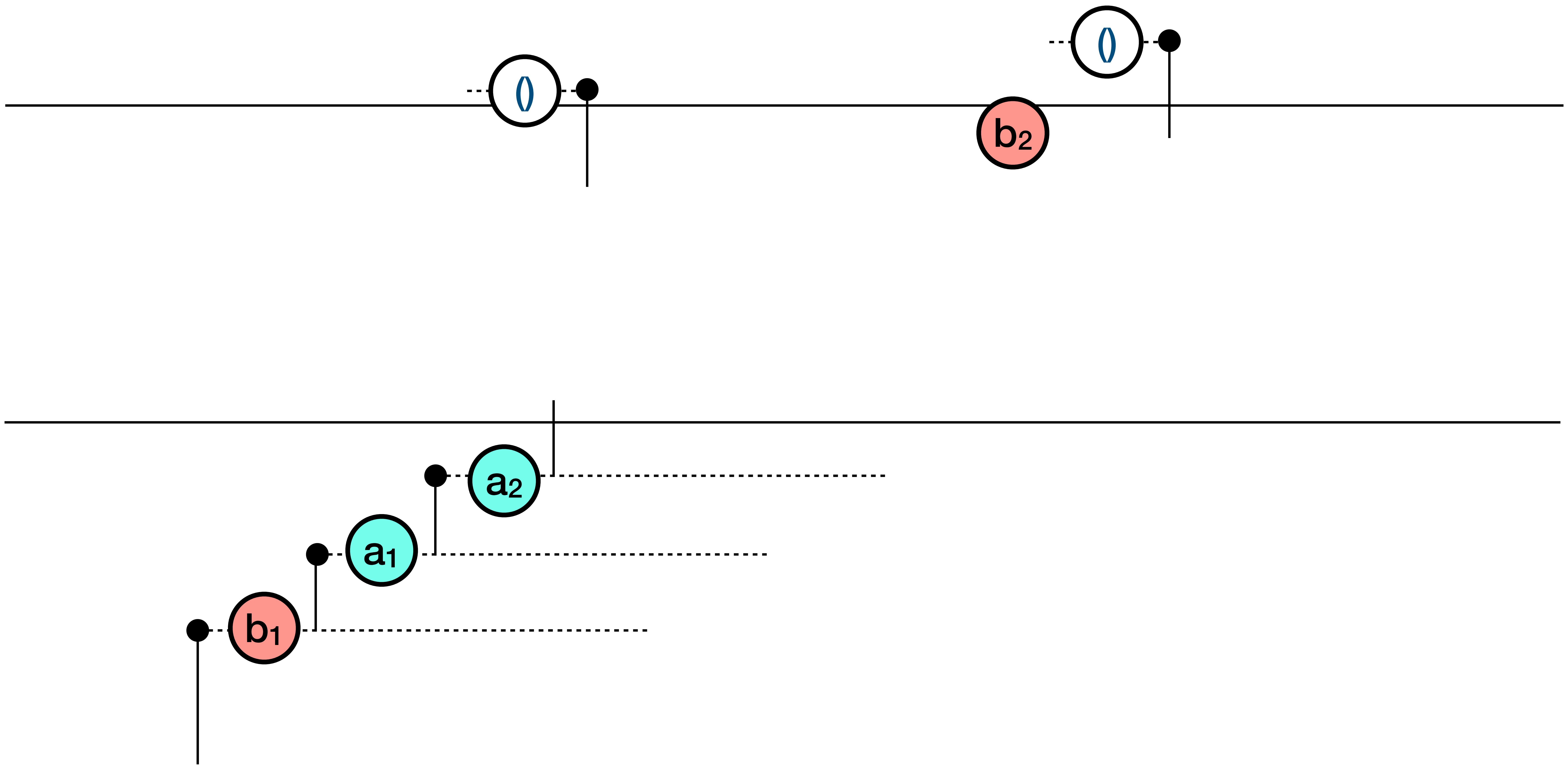

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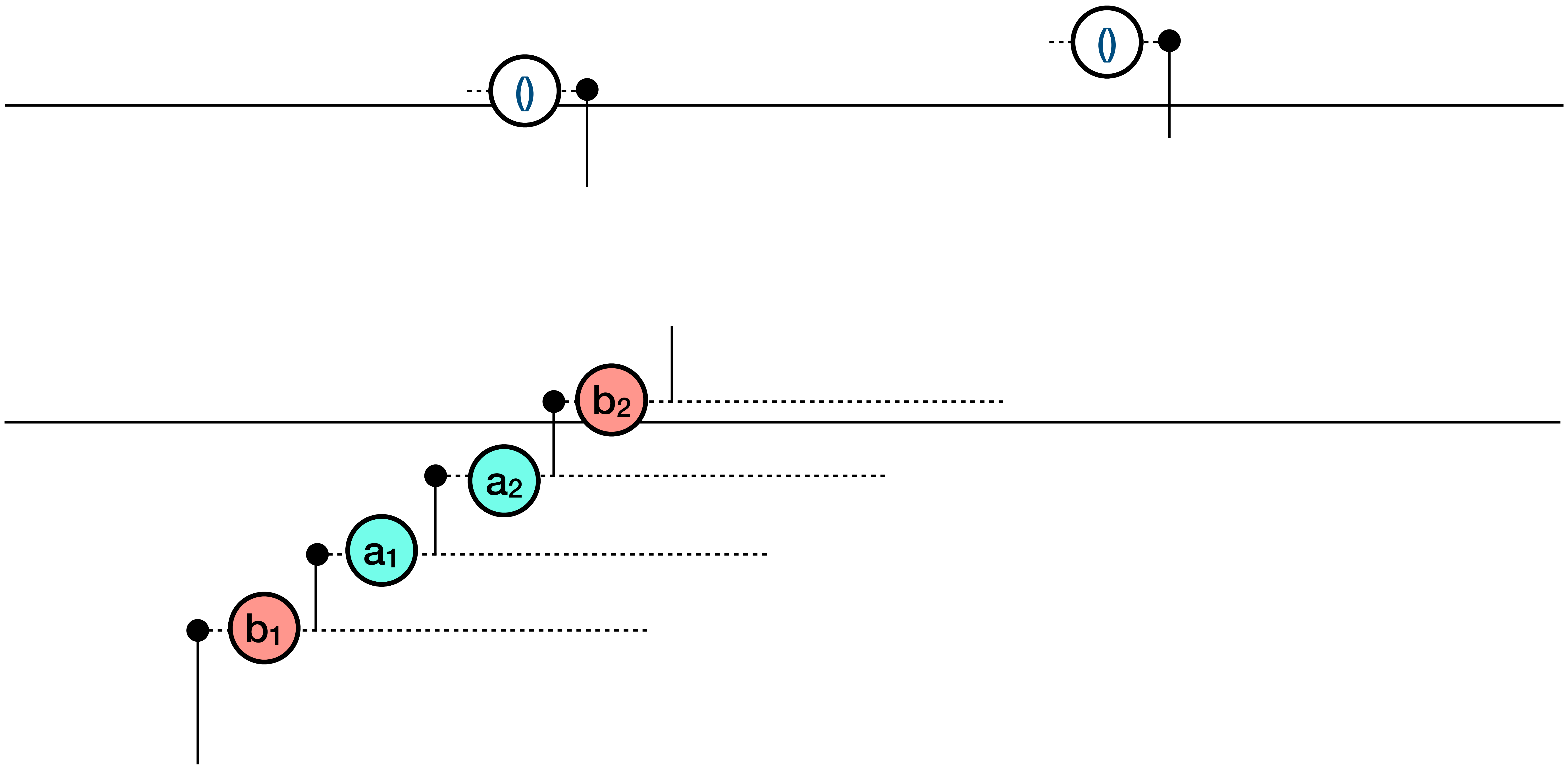

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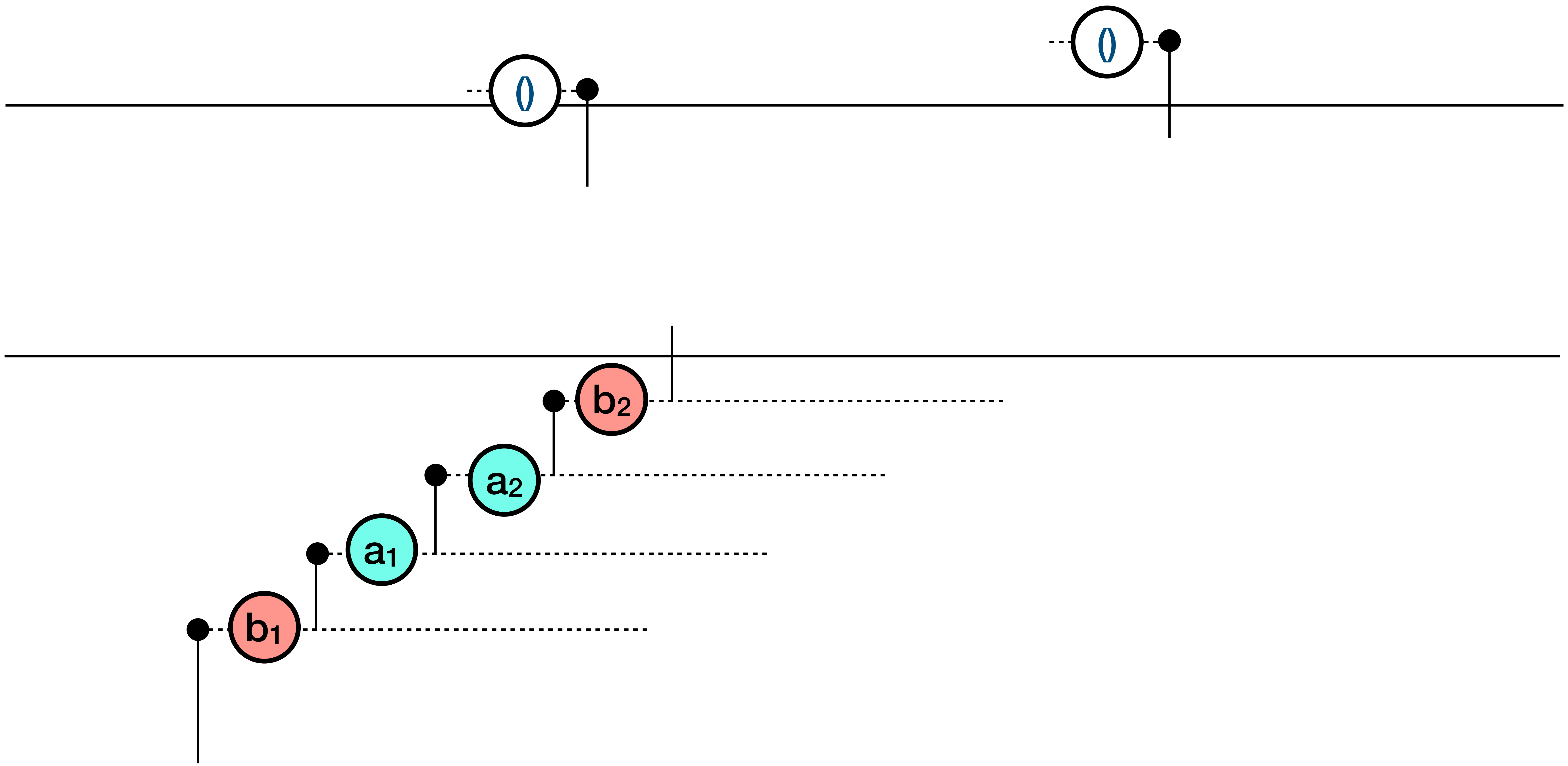

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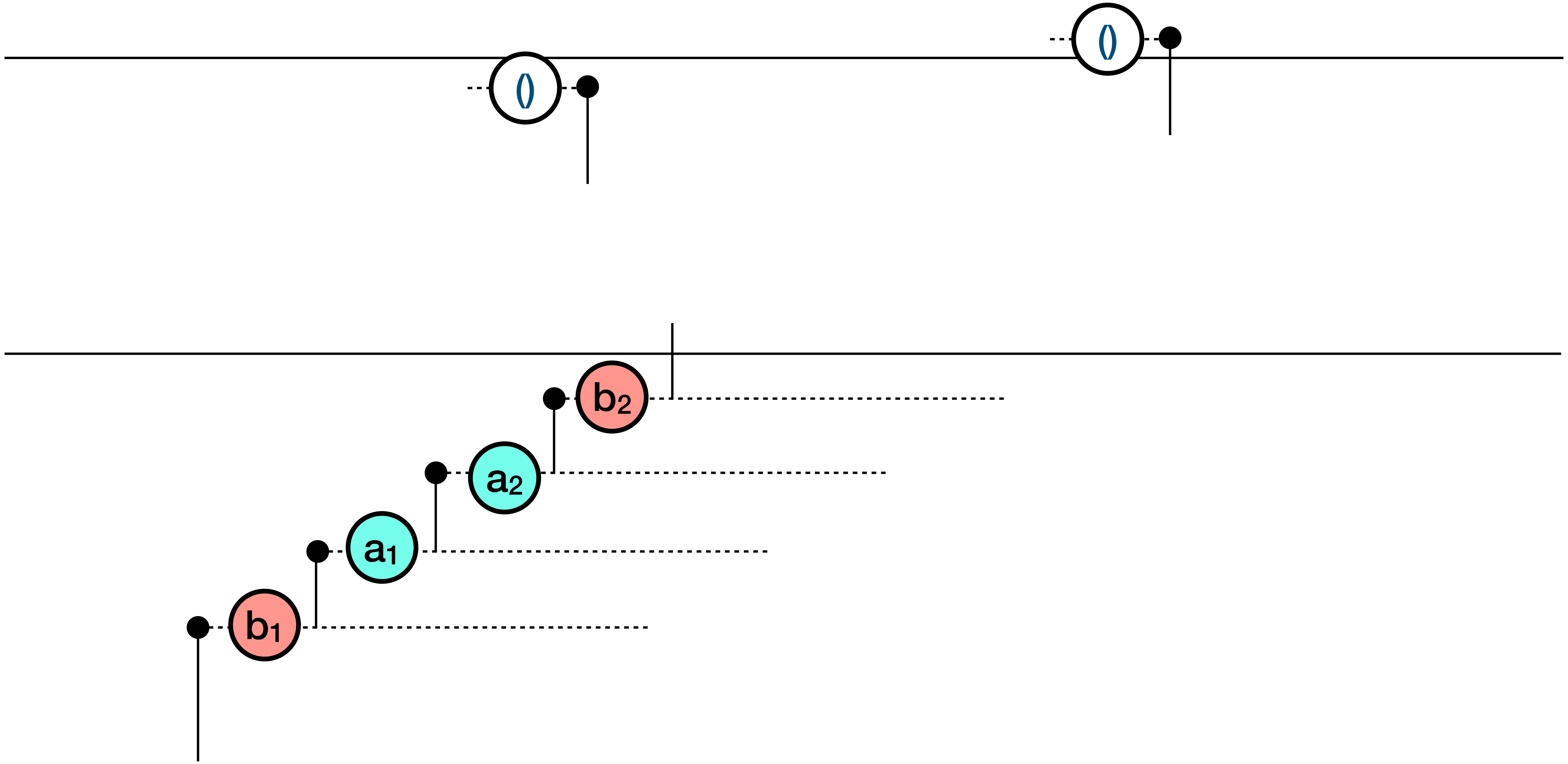

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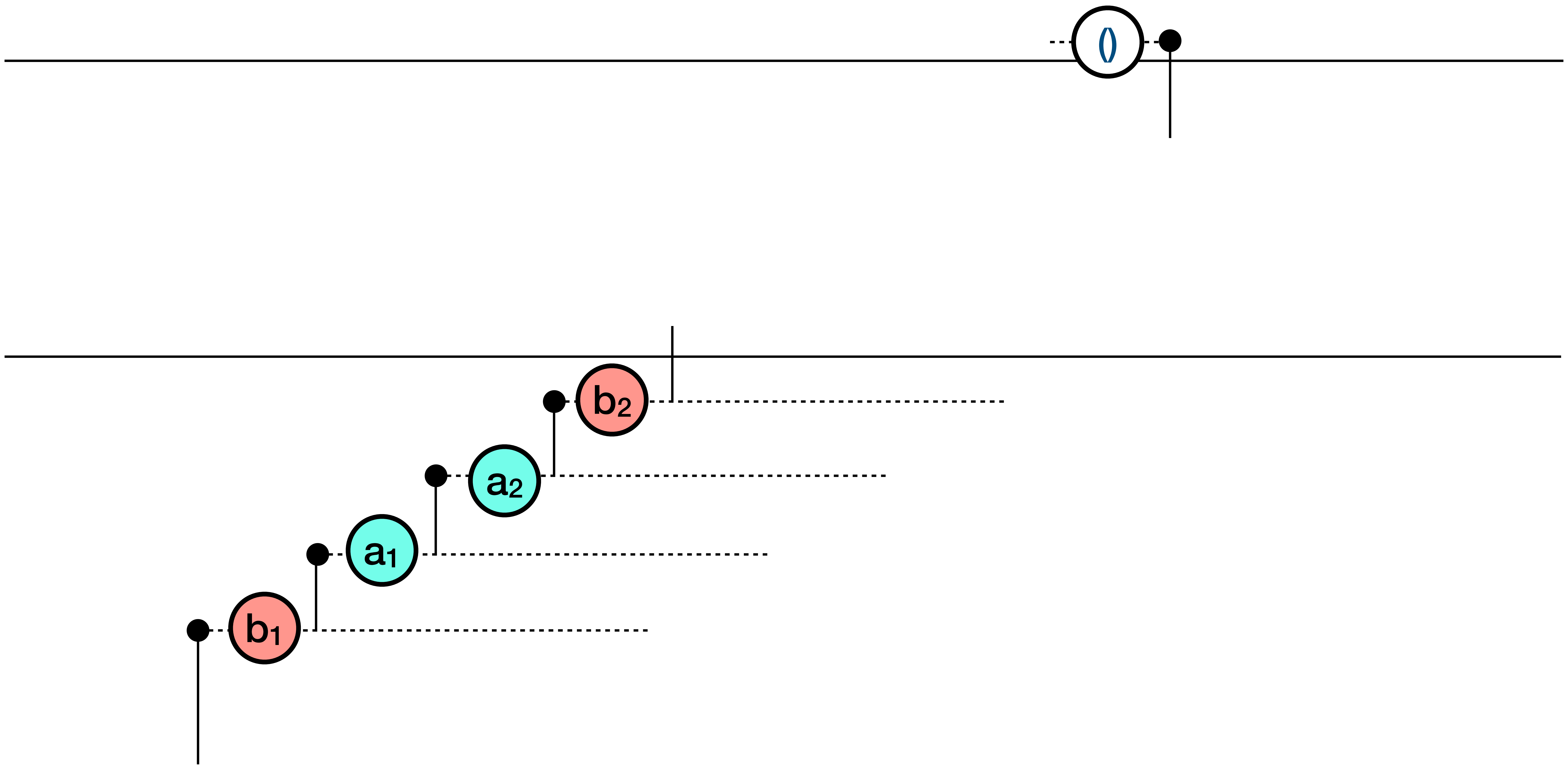

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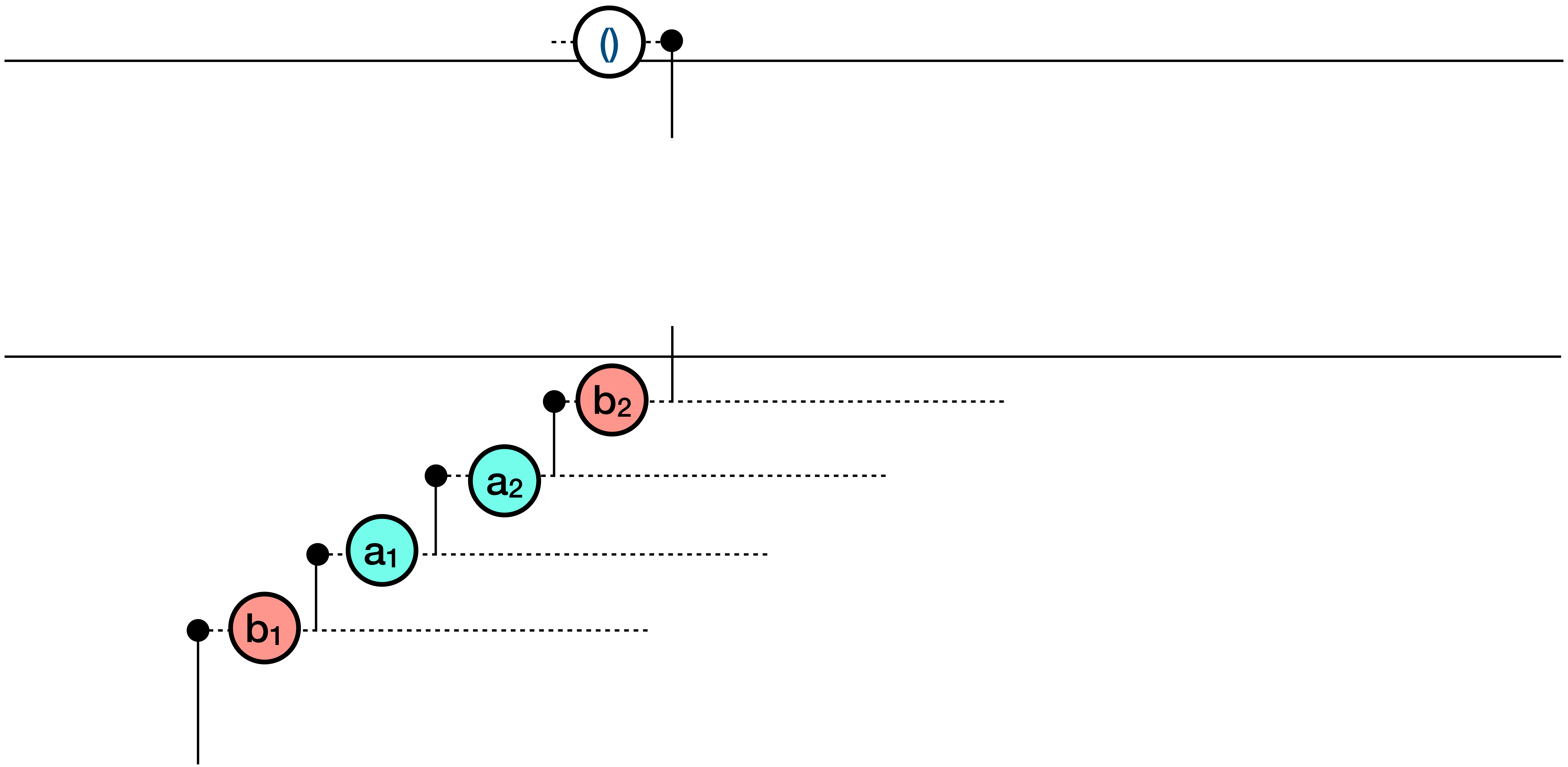

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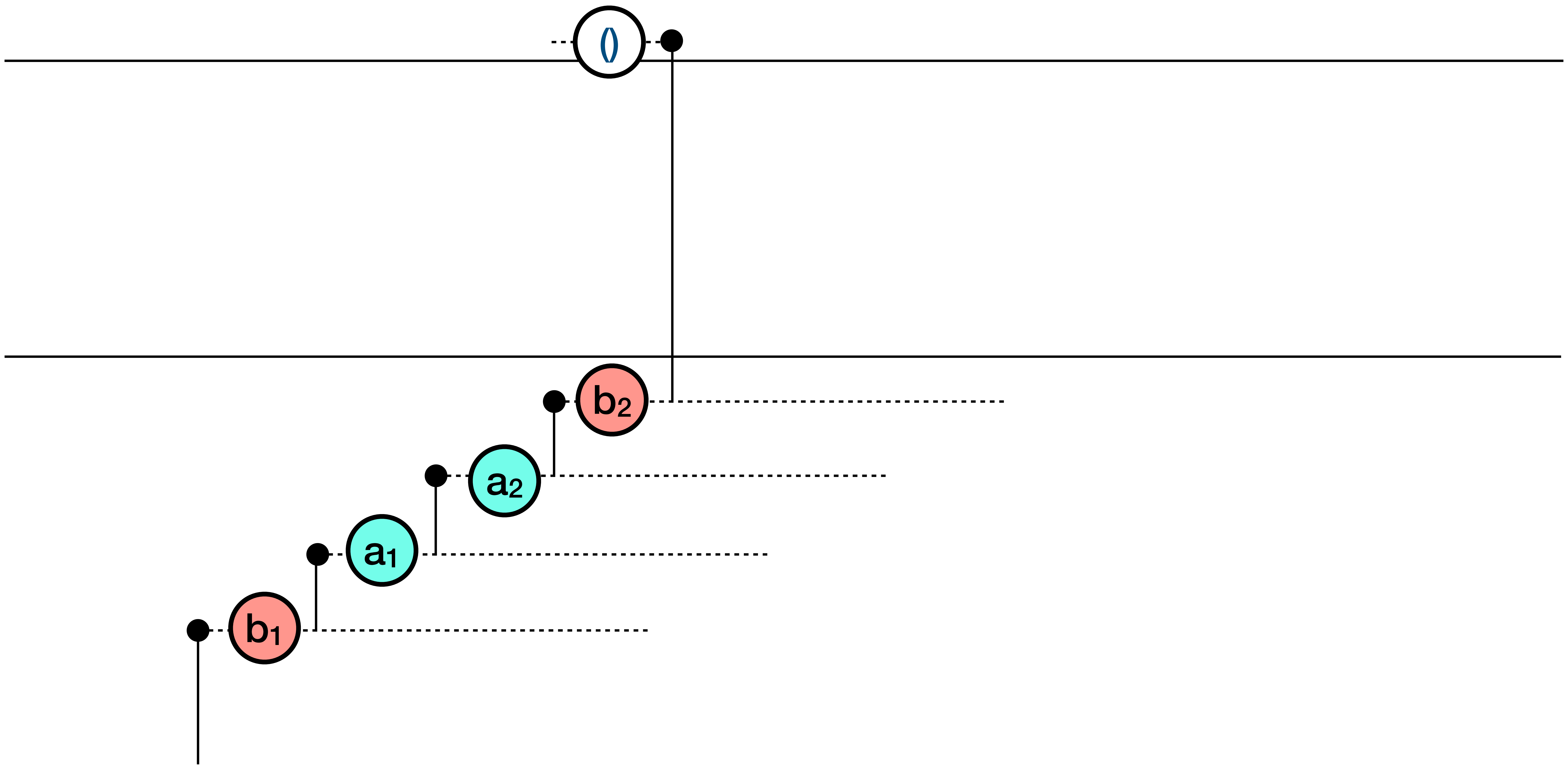

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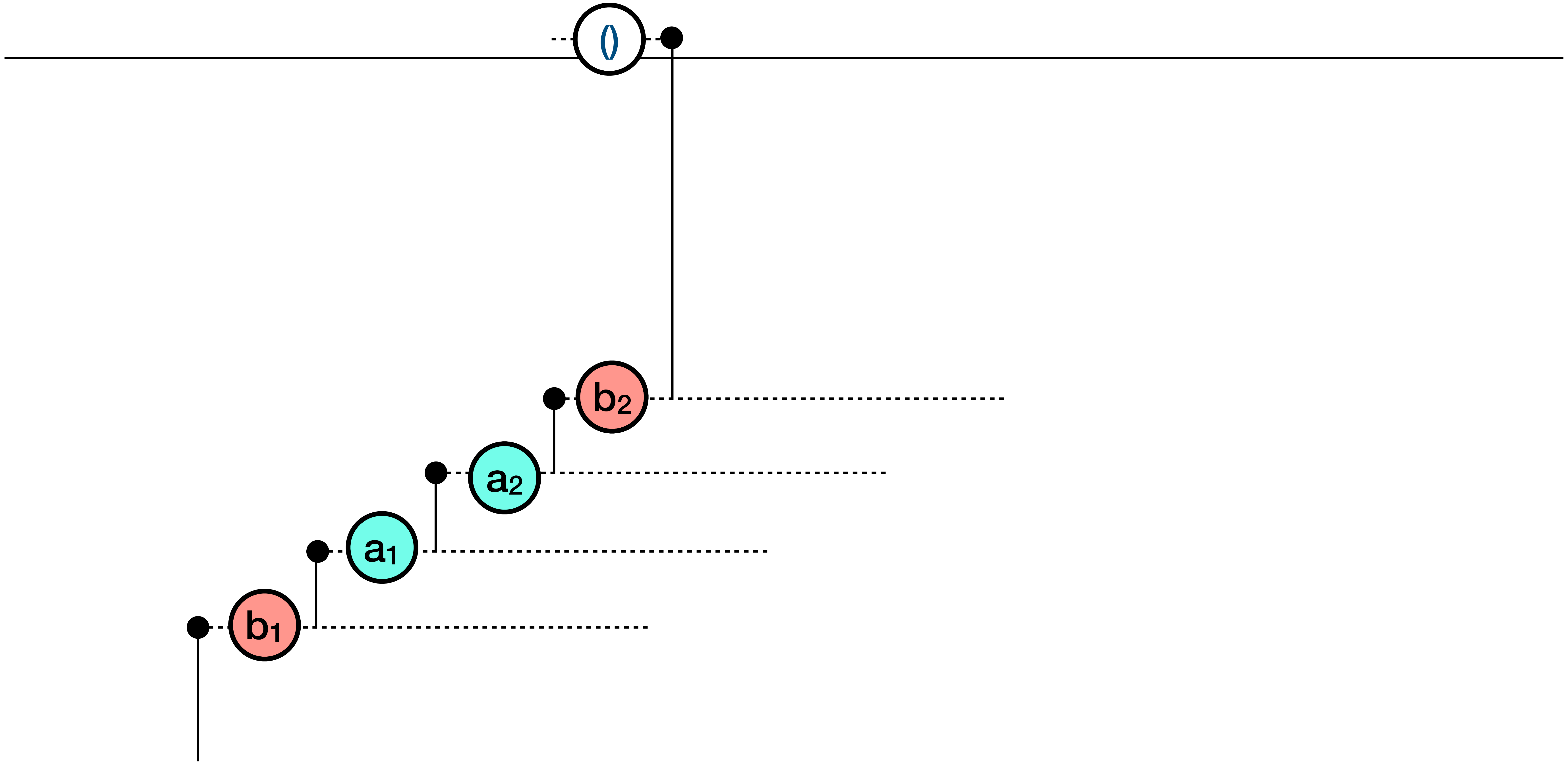

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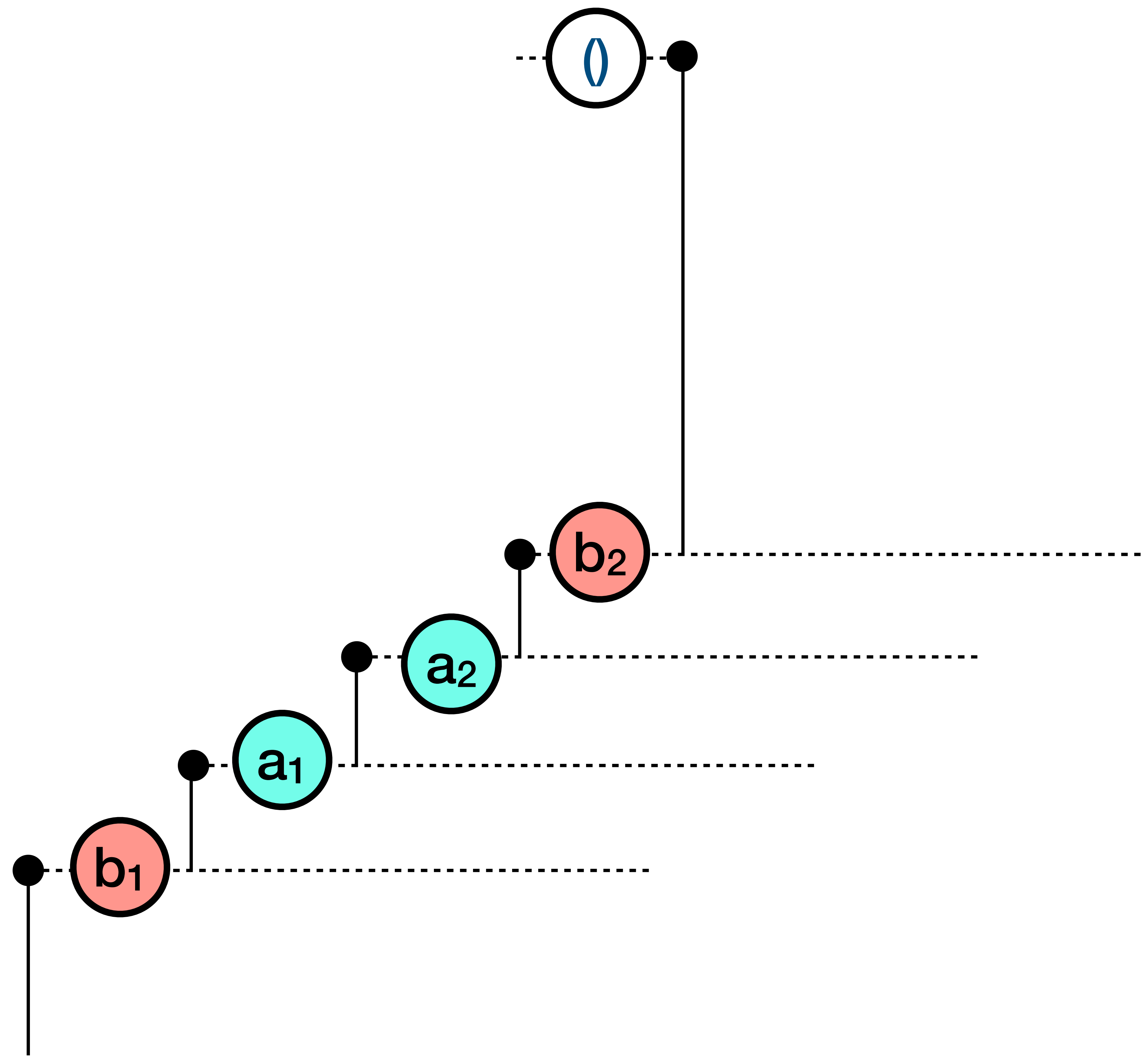

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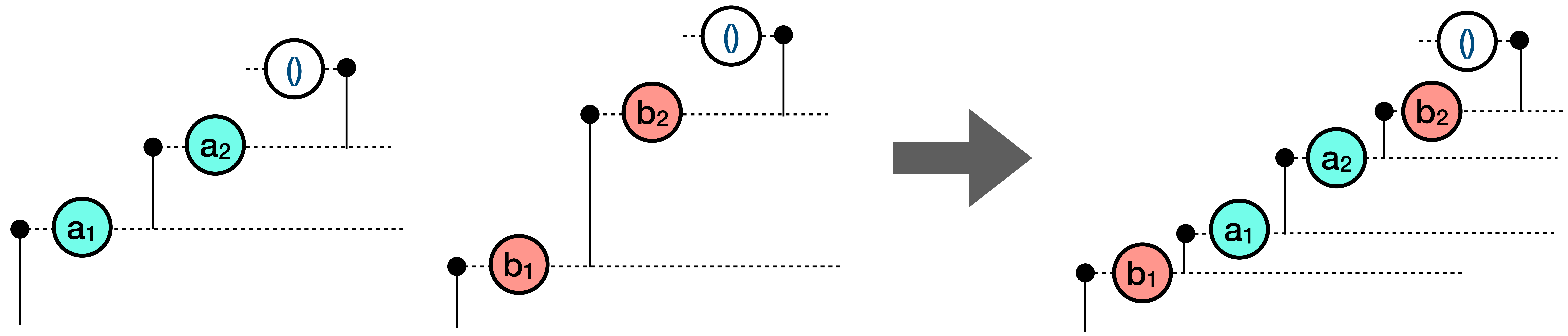

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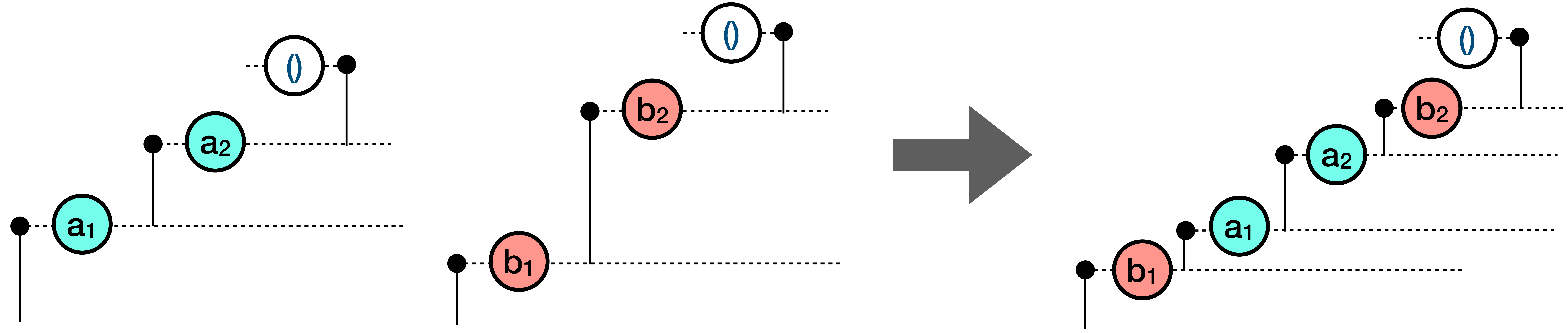

Preserves *"temporal"*(*) order

(*) but there's no global time

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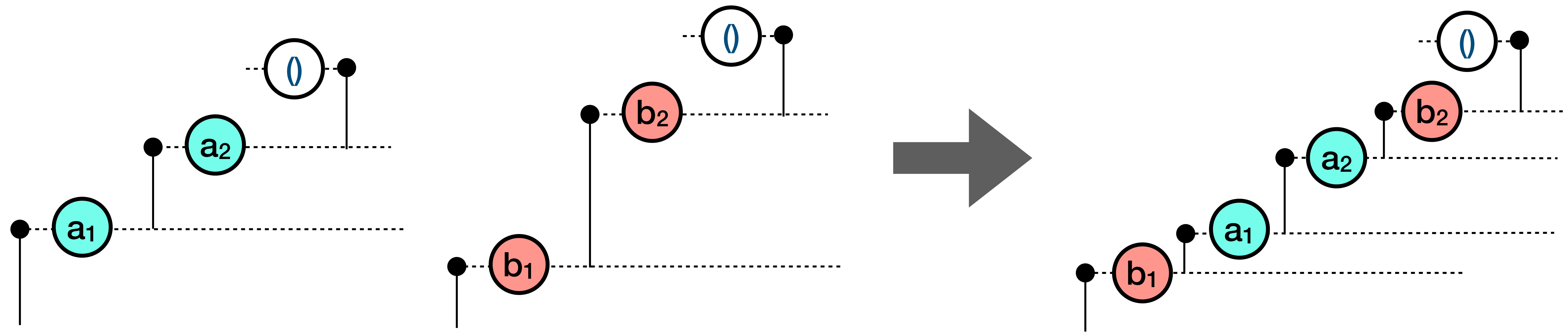

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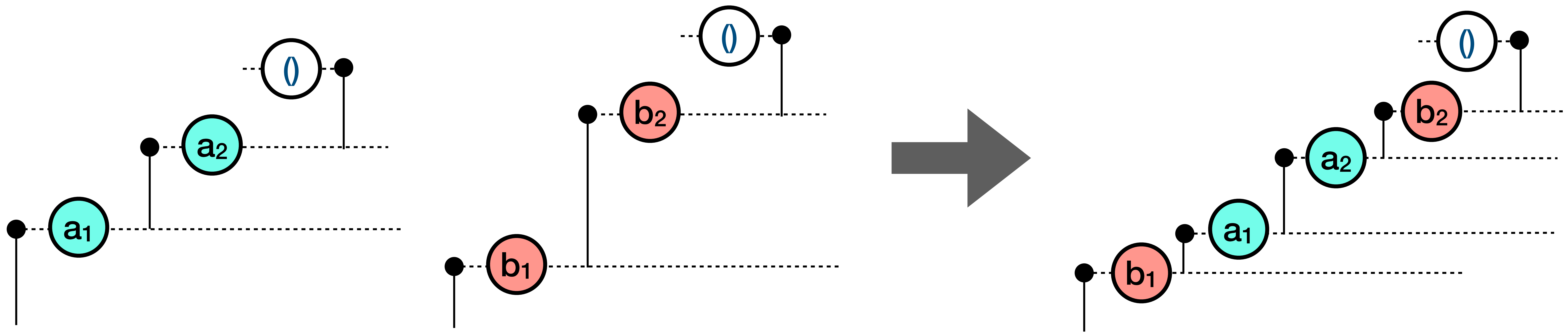

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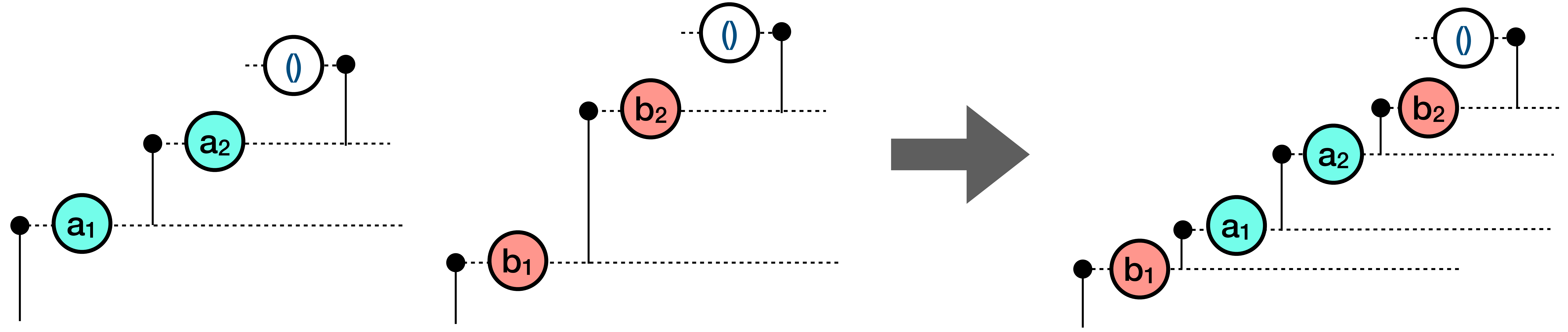

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It seems so, but what about the **non-determinism?**

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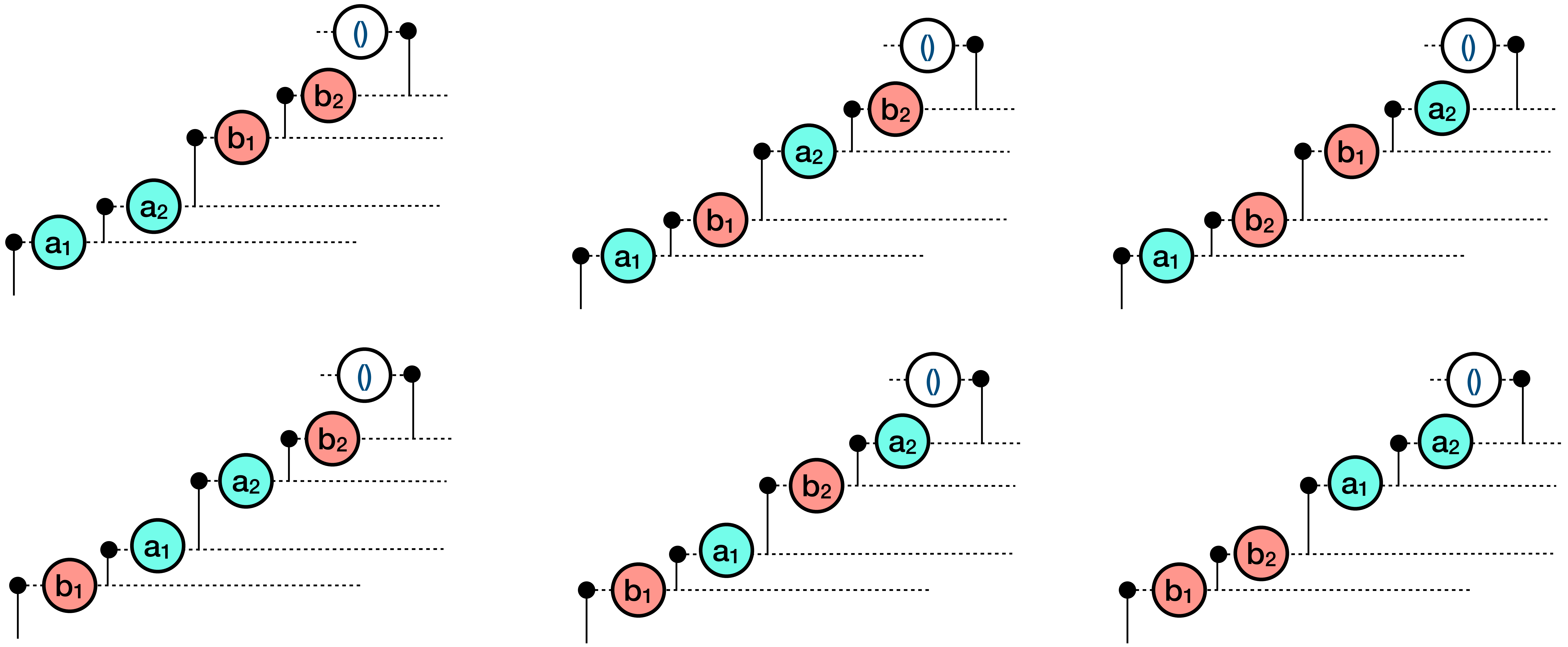

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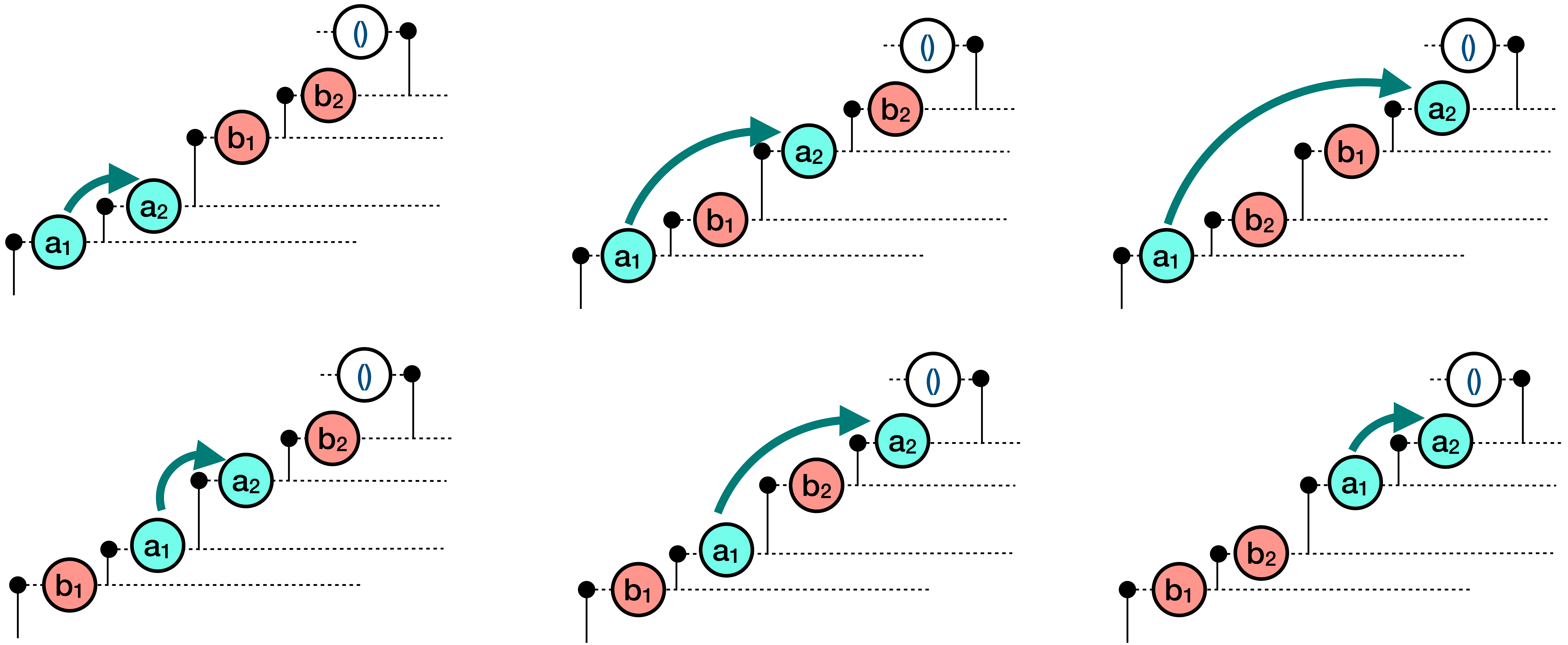

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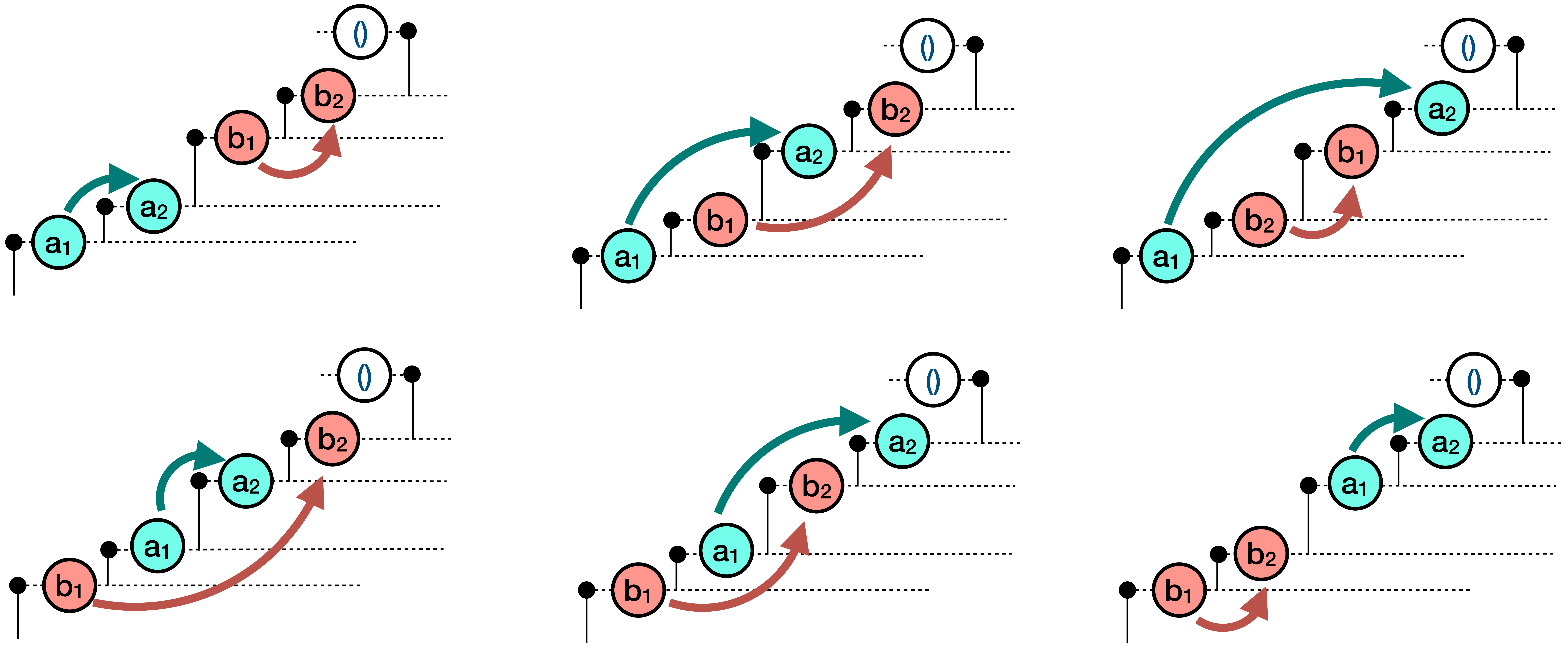

Equality of Non-deterministic Functions

`f = g`

What does it mean for **non-deterministic** *f, g?*

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Non-deterministic functions

- *"same **set** of observable behaviors"*
- not necessarily the same probabilistic distribution of them

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1. Consider any observable behavior of the left side.
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3. And vice versa.

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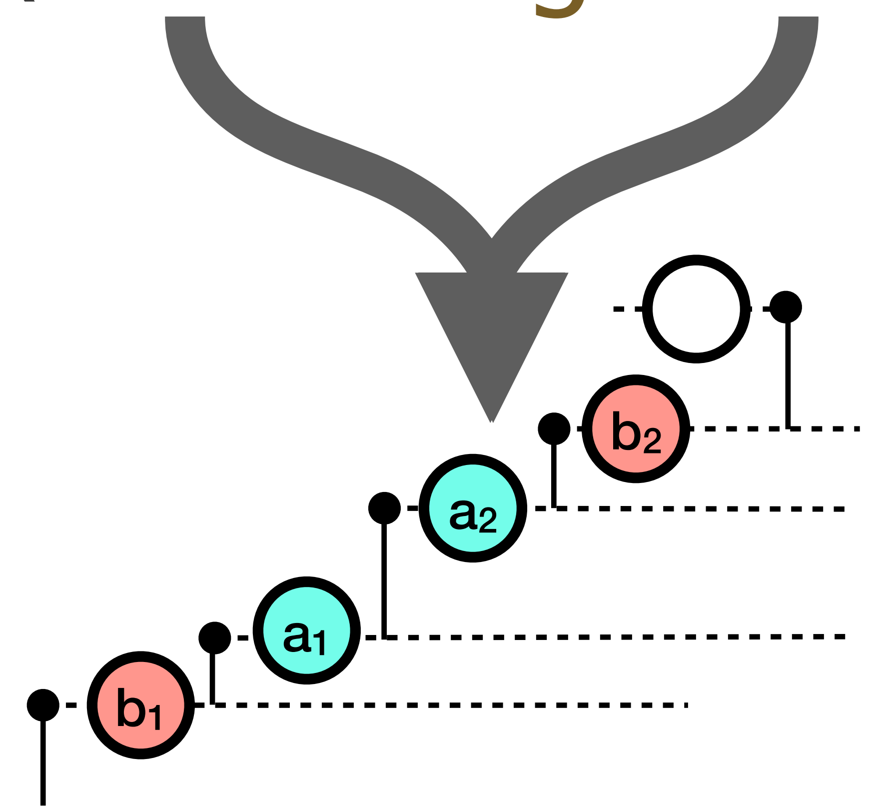

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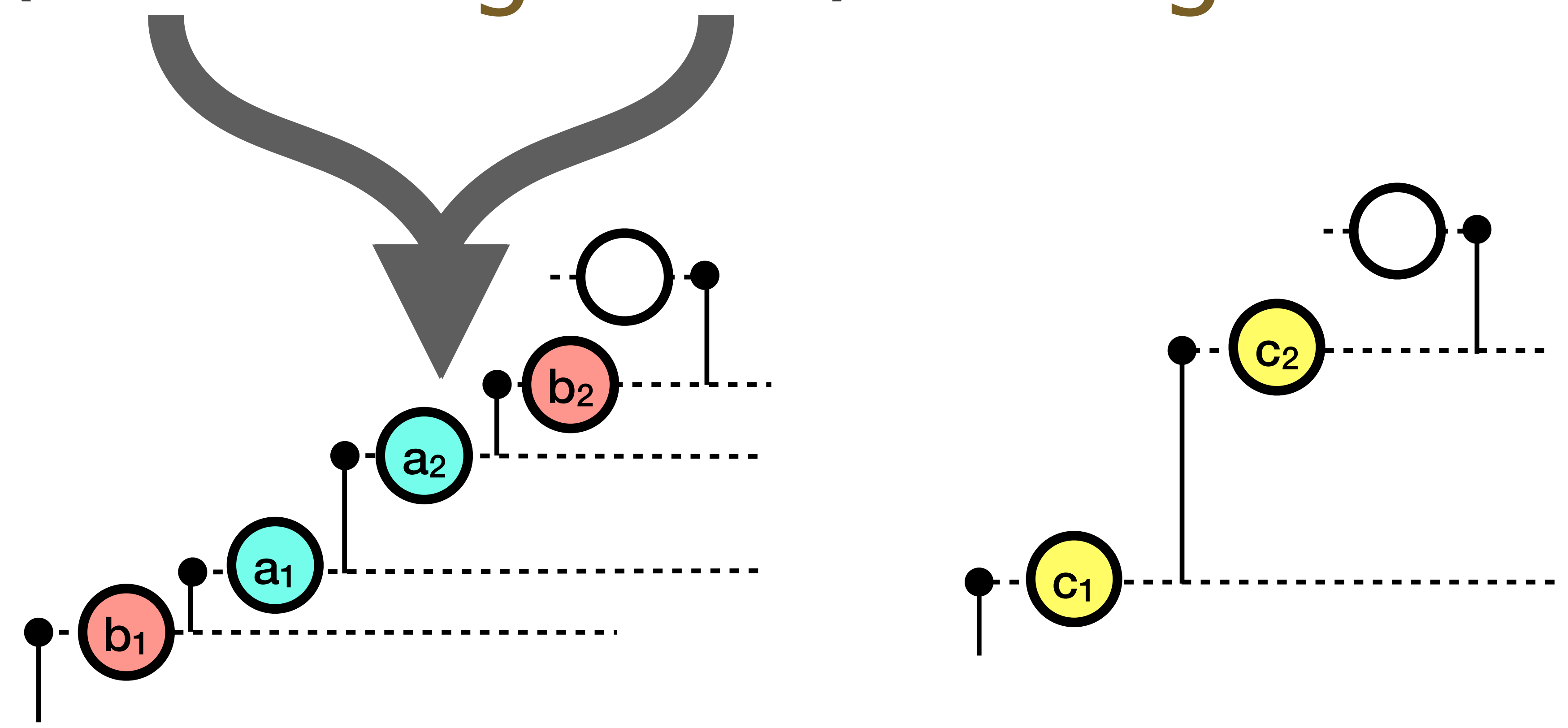

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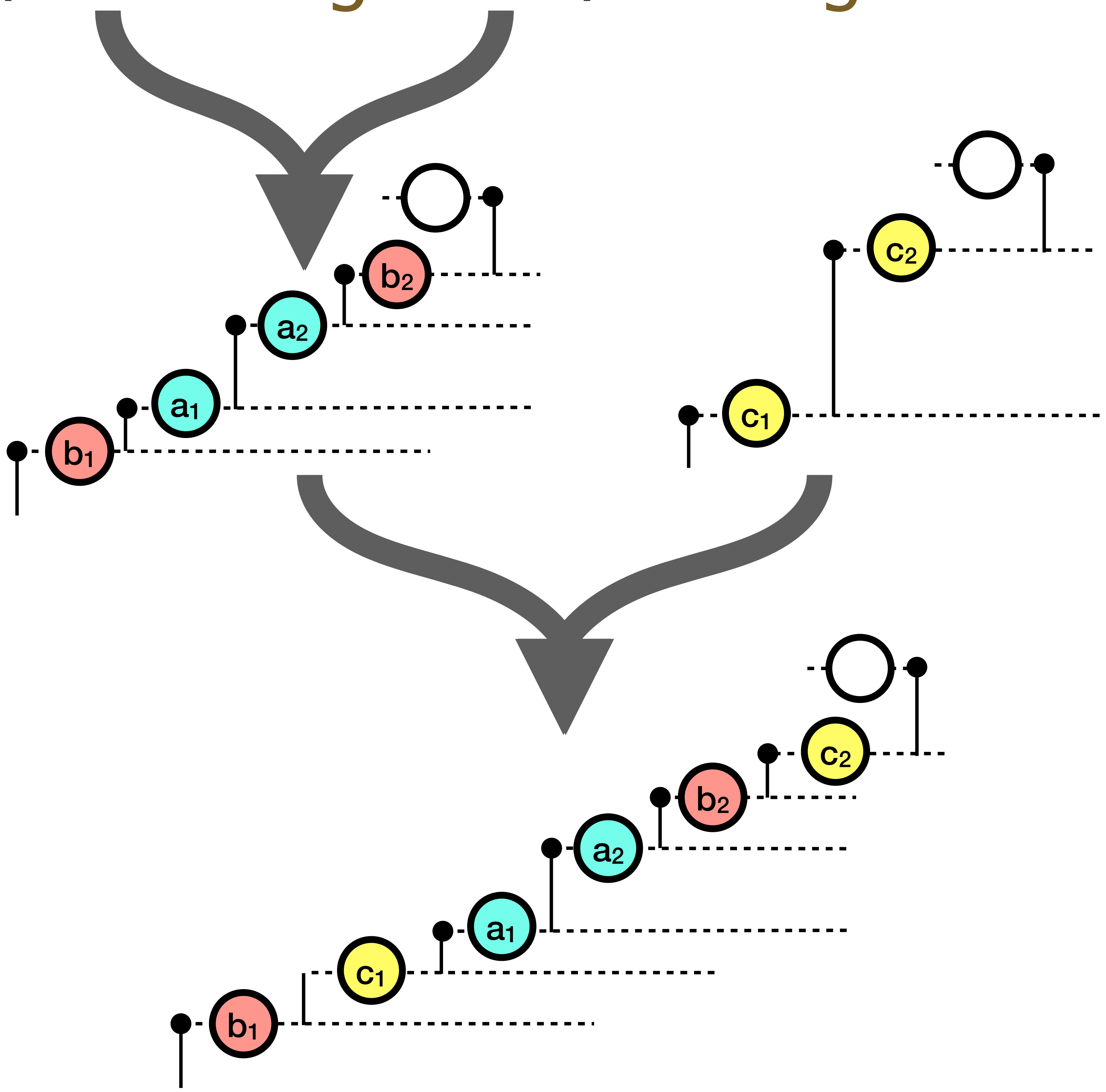

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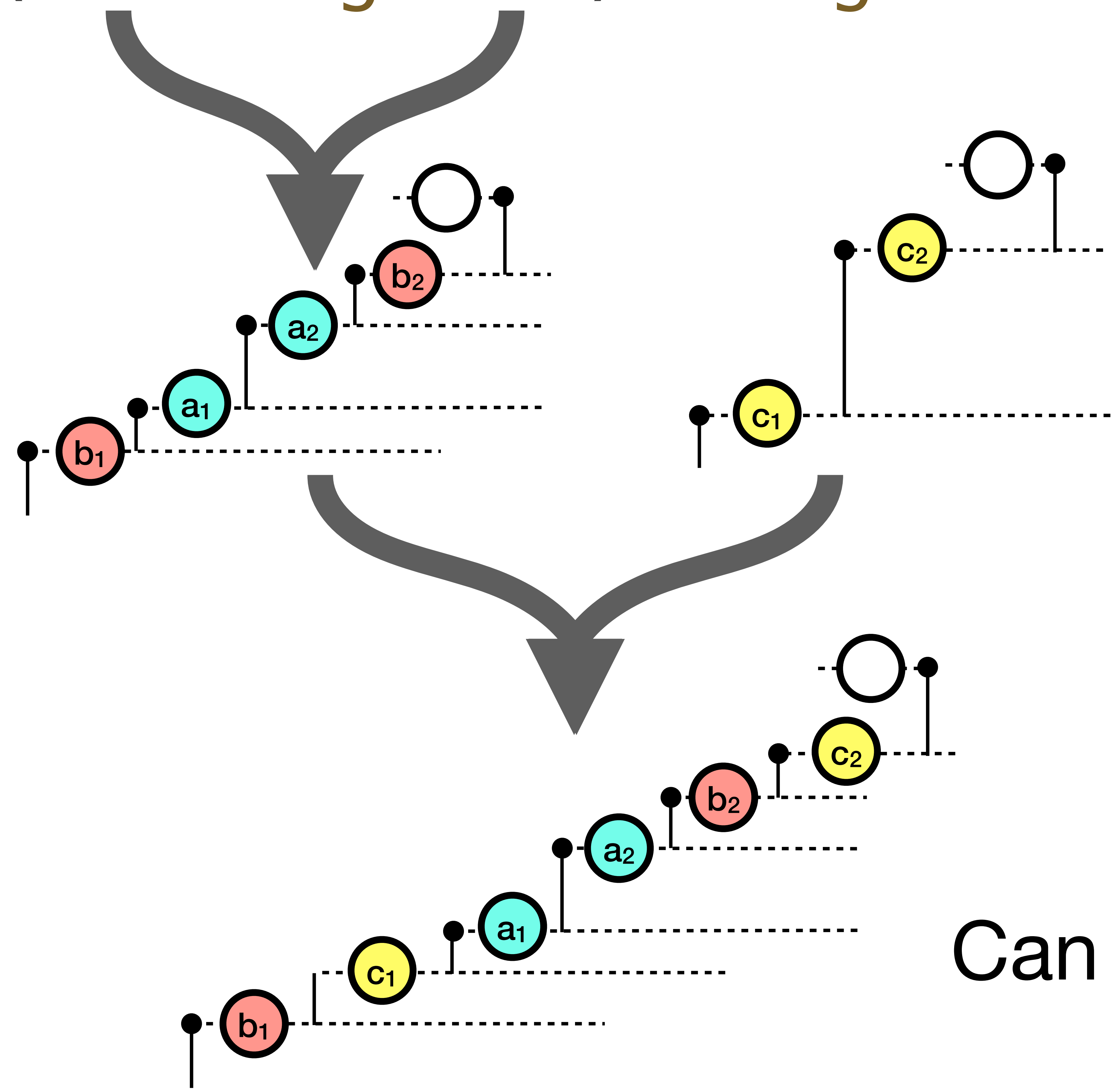

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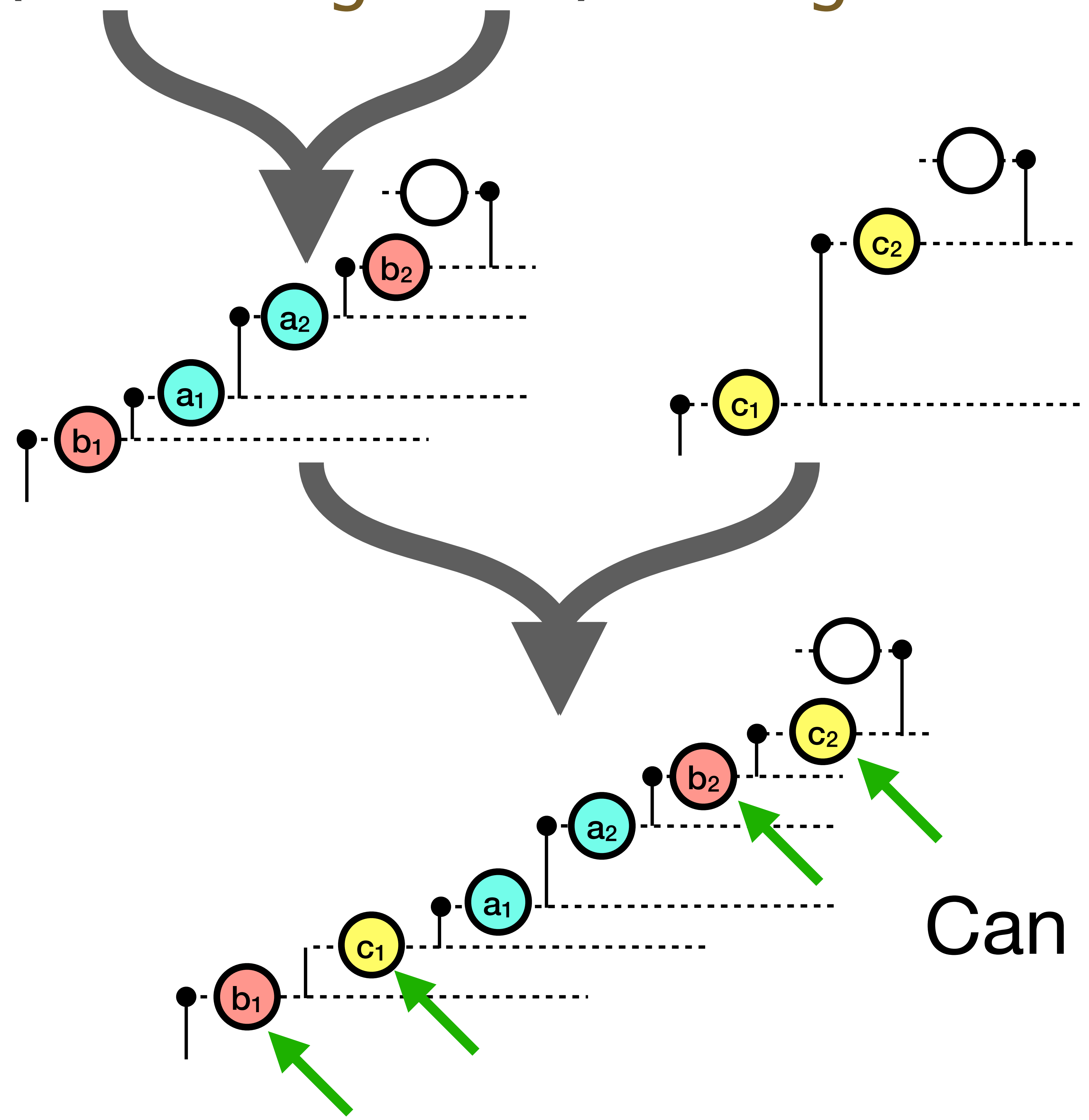

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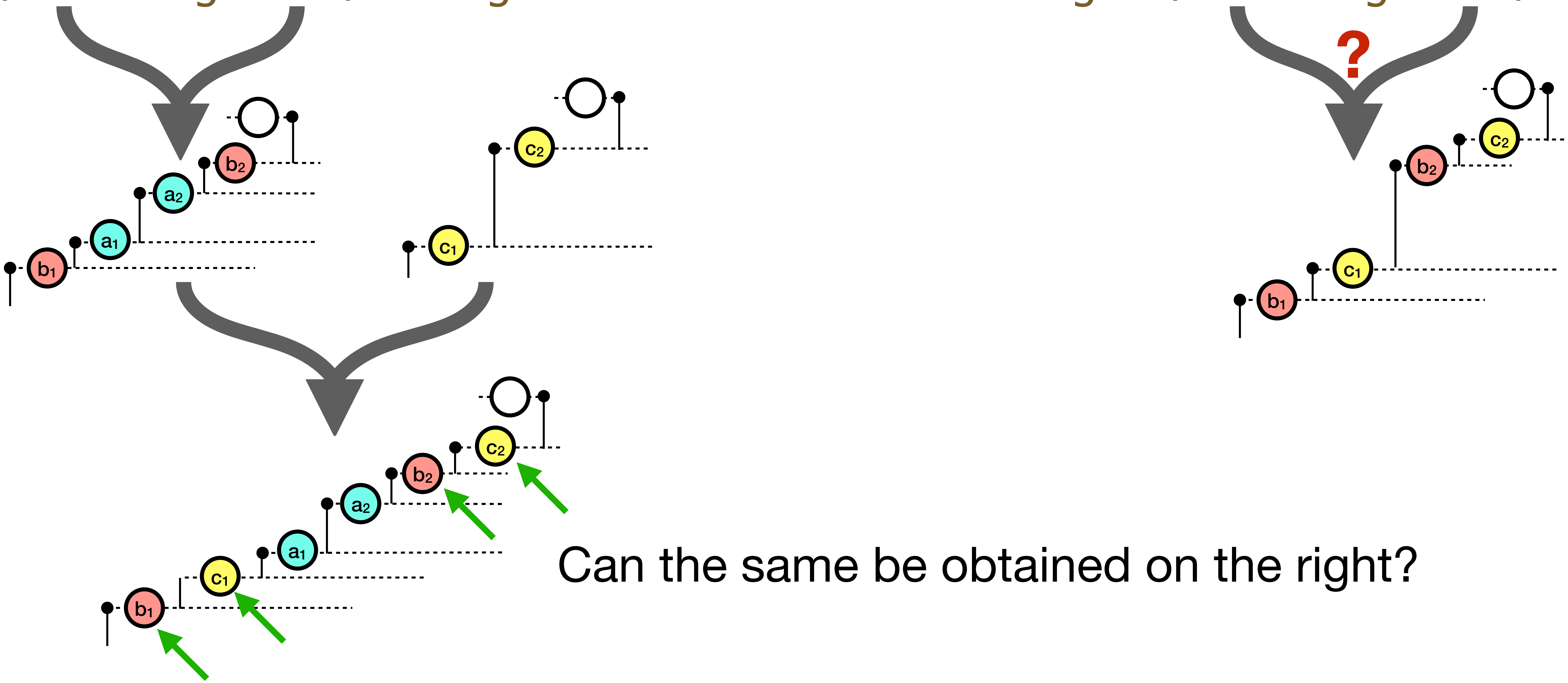

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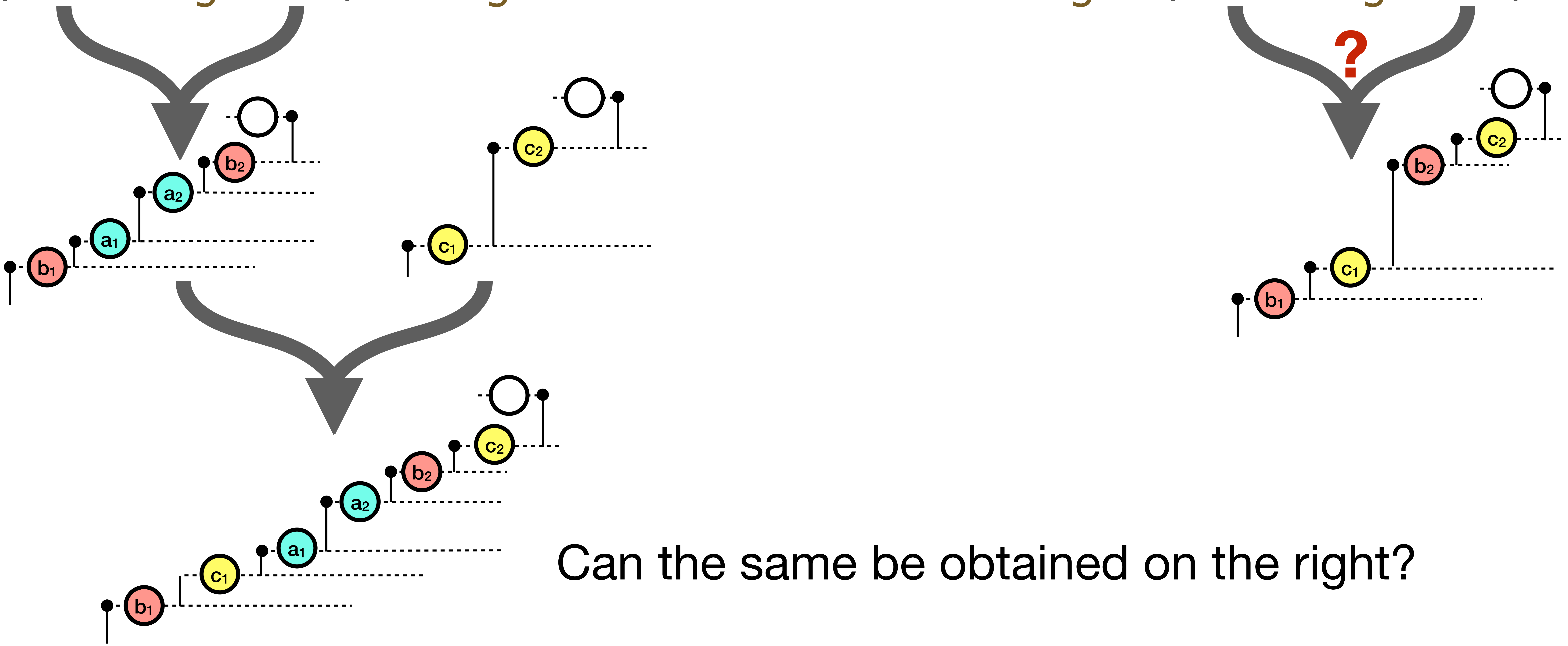

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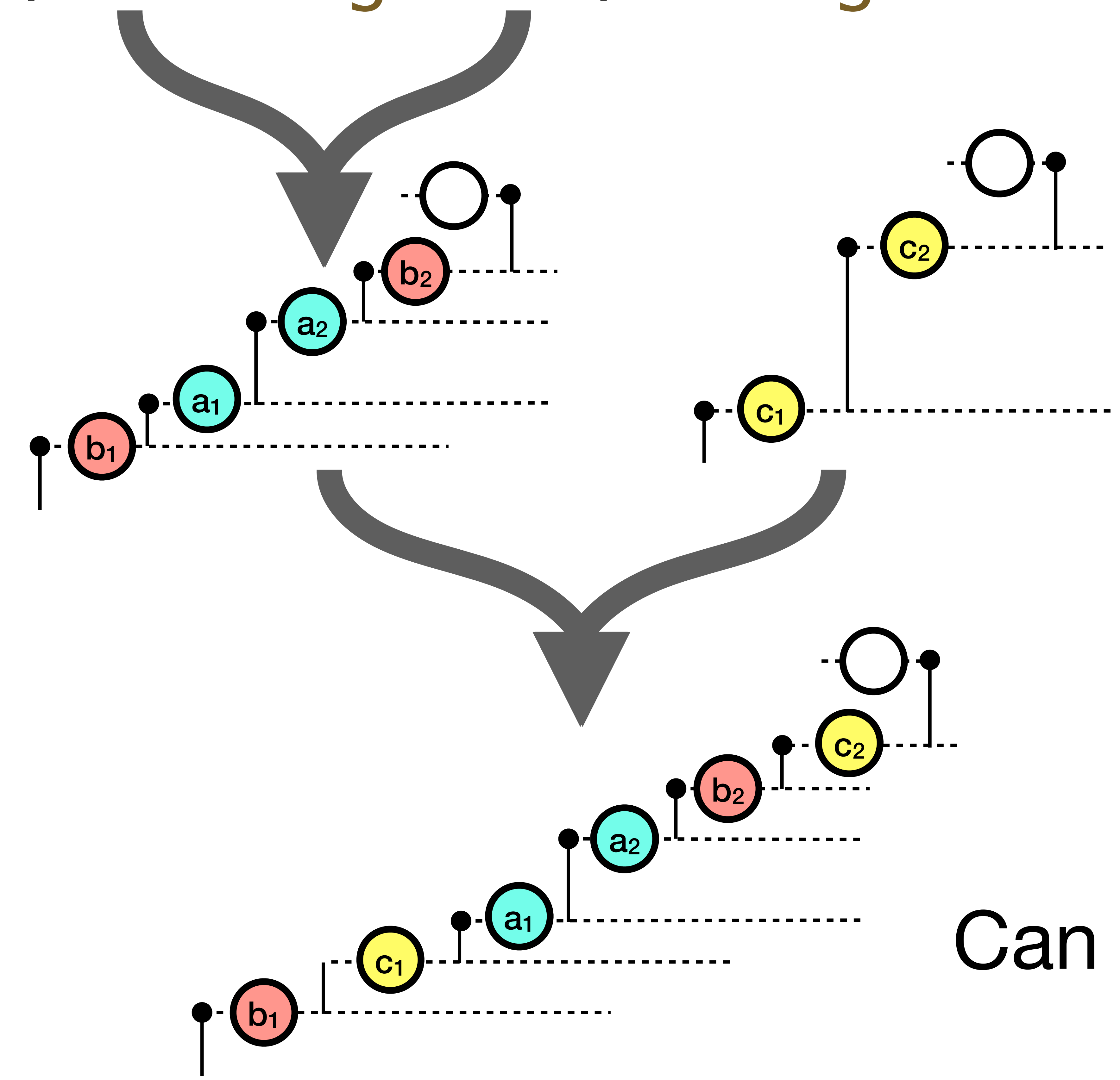

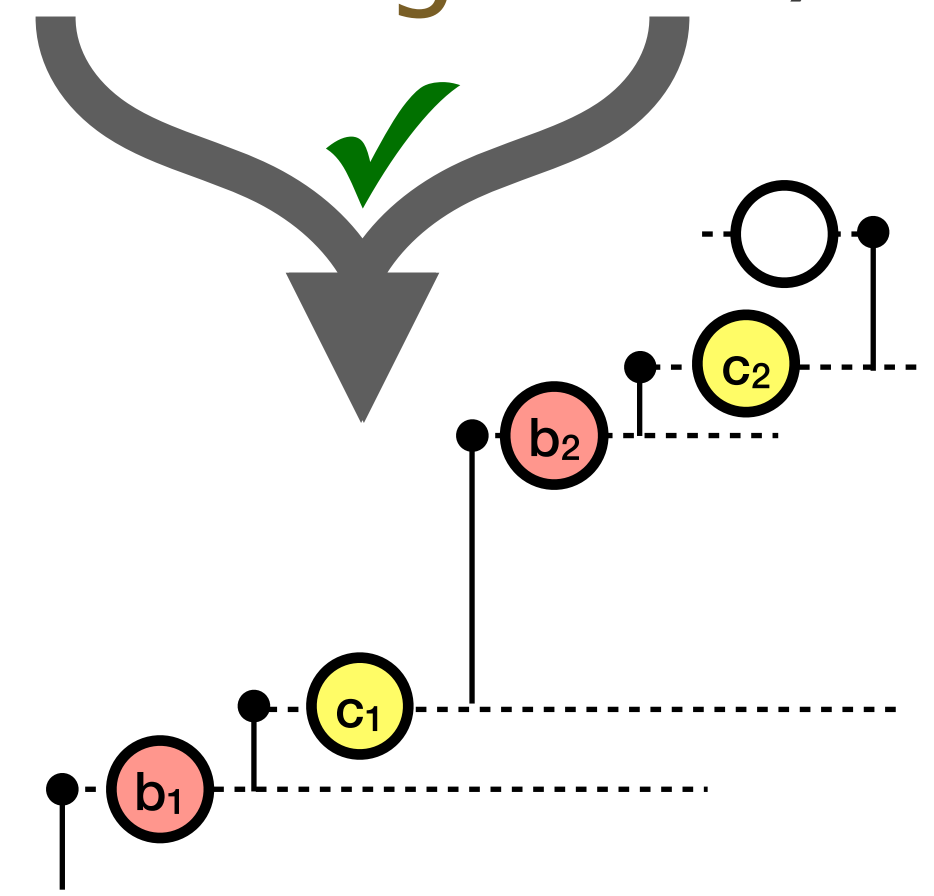

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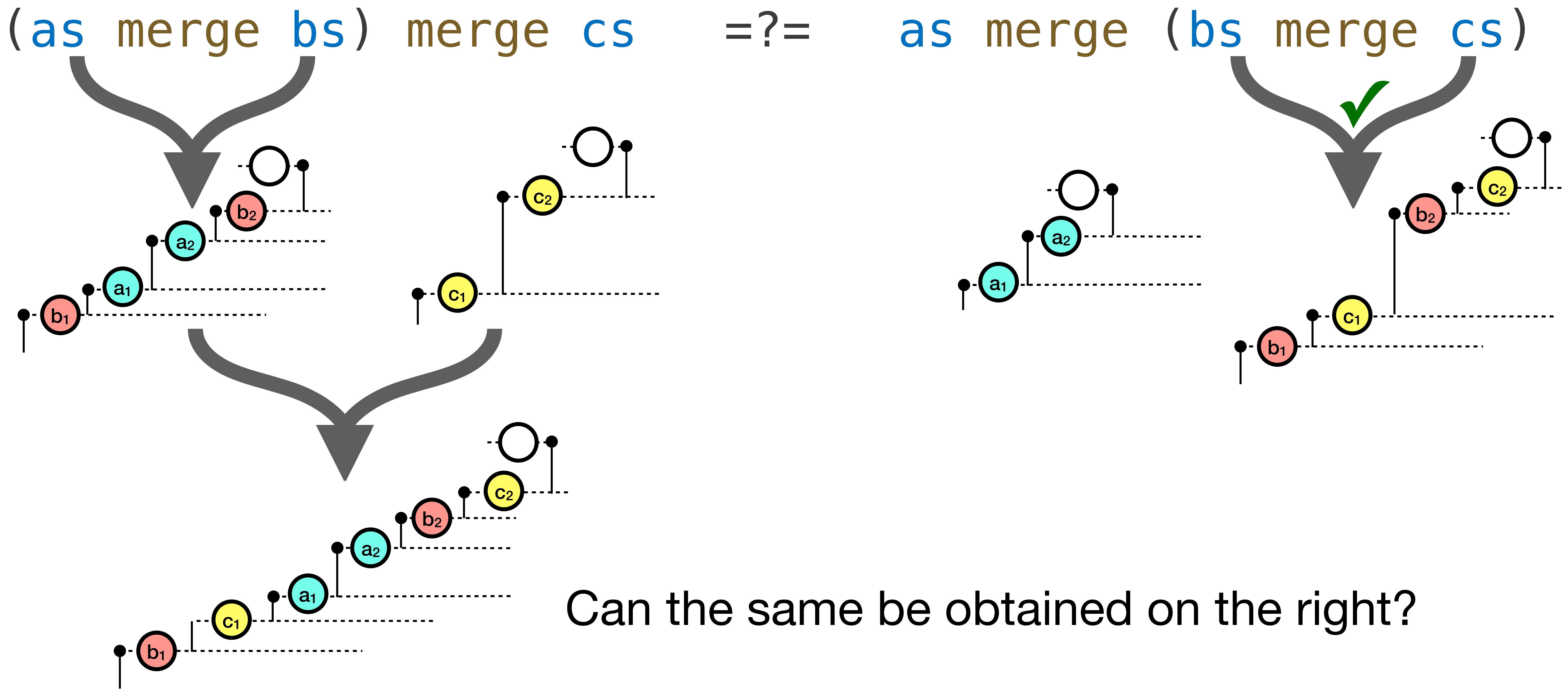

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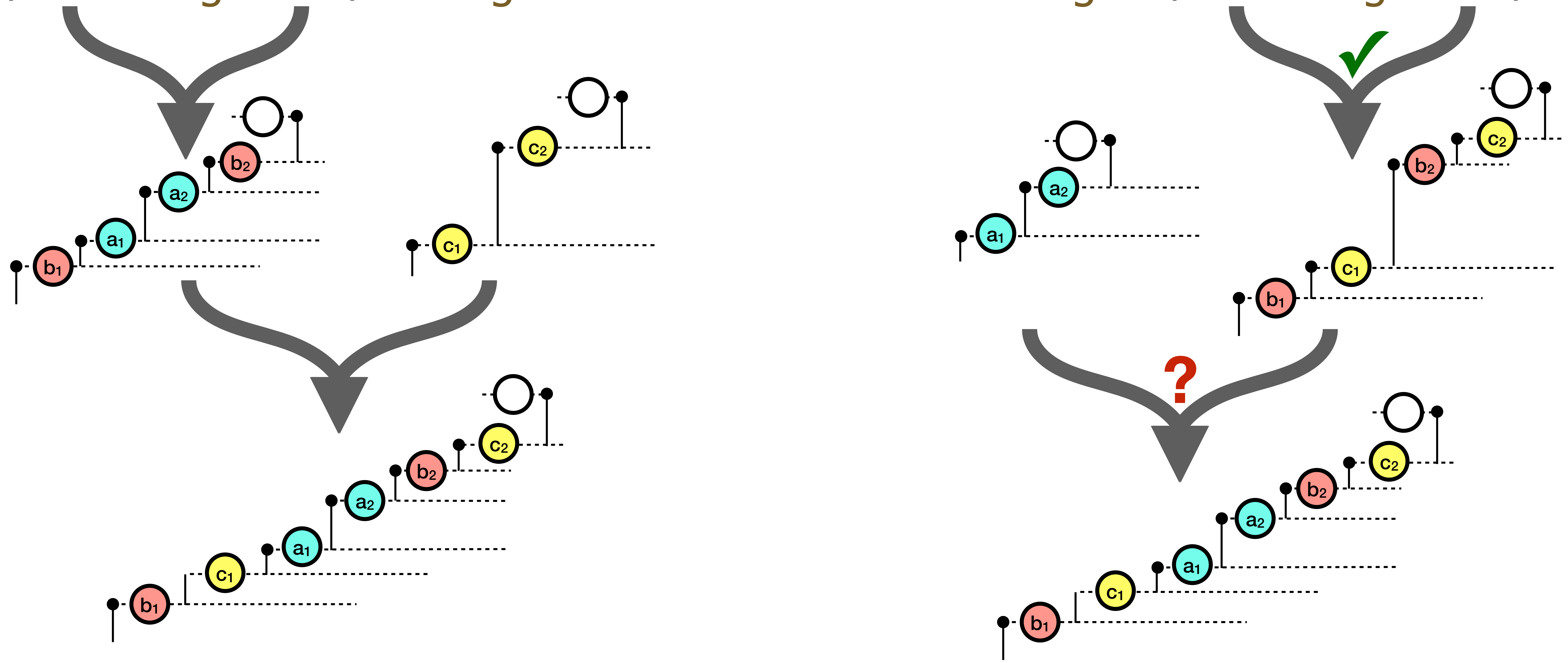

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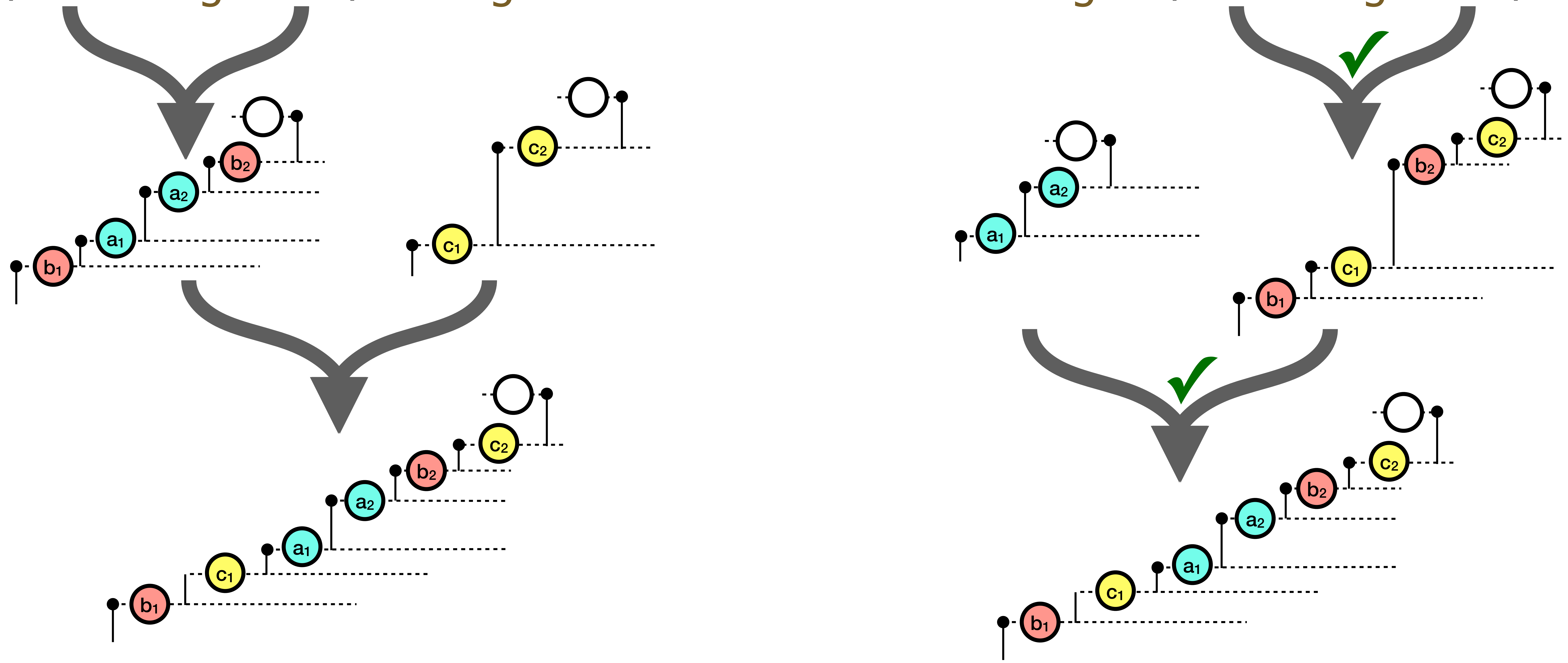

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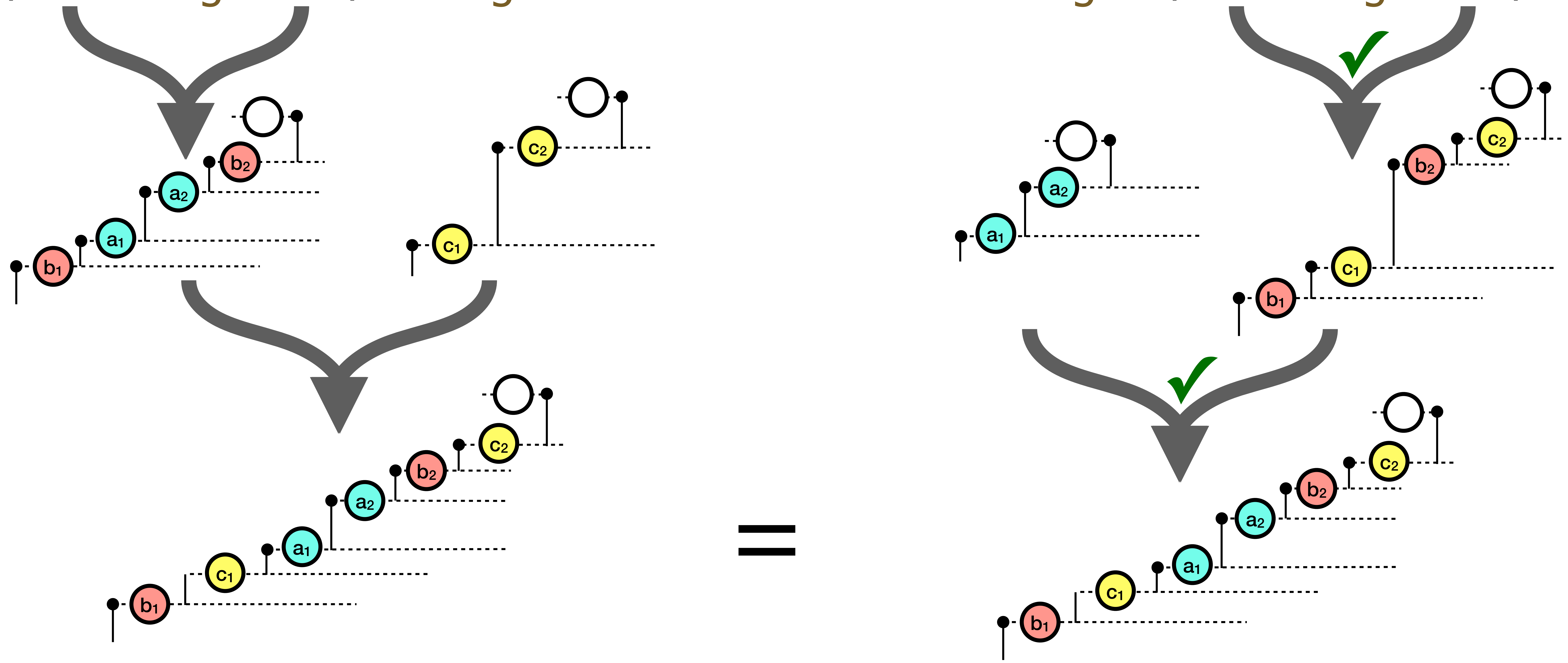

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type Writer[A] = List[String] ⊗ A
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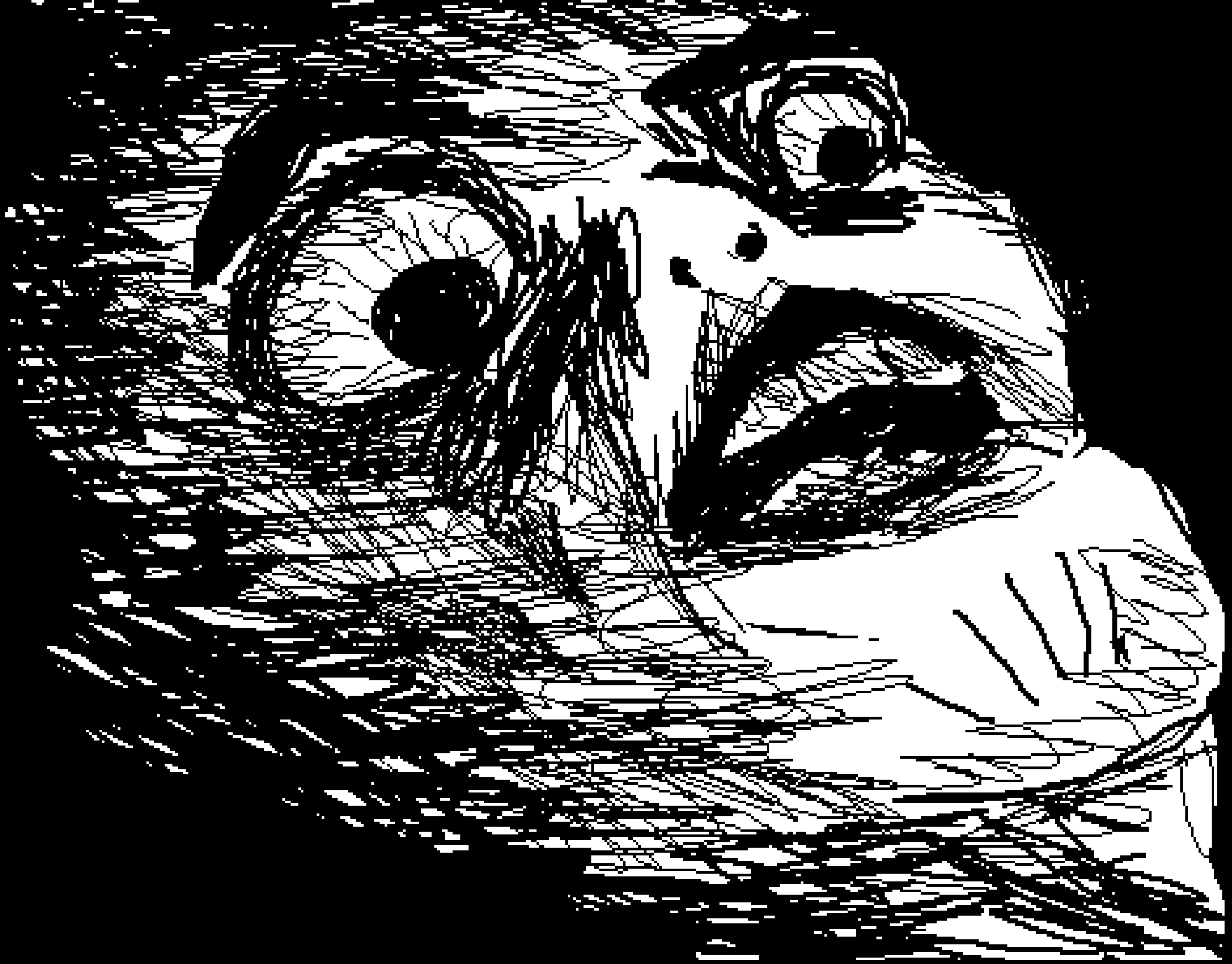
Where's the sequencing?



THERE IS NO SEQUENCING

Lessons

- Monads definable in **any** Category (even non-executable one, like **`<:<`**)
- **Syntactically,** monads *do* support **sequential composition**
- Sequential composition ≠ sequential **execution** (e.g. monads in **`<=`**)
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- Sequential composition ≠ sequential **execution** (e.g. monads in **<=**)
- *"Sequencing of effects"* is **vague,** definable only **tautologically**
- There exist **lawful monads** with **non-deterministic** behavior

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- **What else are we wrong about?**

Thank You!

Scala examples: https://github.com/TomasMikula/non-sequencing-monads/

Libretto: https://github.com/TomasMikula/libretto/